MICHAEL ROBINSON: RESEARCH STATEMENT

1. Background

1.1. **Specification of the problem.** My work concerns the classification of global solutions to

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + \sum_{i=0}^{N} a_i(x) u^i(t,x), \tag{1}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and the a_i are bounded and continuous. By a global solution, we mean one that is defined for all t and x and is continuous. This kind of equation provides a simple model for a number of physical phenomena.

For instance, choosing the right side to be $\Delta u - u^2 + a_1 u$ results in an equation which can be thought of as a model of the population of a single species with diffusion and a spatially-varying carrying capacity, $a_1(x)$. It can also be thought of as a toy model of combustion. It is well-known that if a_1 is a positive constant, then it supports traveling waves. Such traveling waves can model the propagation of a flame through a fuel source.

The equation (1) defines the flow of the L^2 gradient of a certain *action* functional,

$$A(f) = \int_{\mathbb{R}^n} \frac{1}{2}\|\nabla f\|^2 + \sum_{i=0}^{N} \frac{a_i(x)}{i+1} f^{i+1}(x) dx. \tag{2}$$

It is then evident that along a solution $u(t)$ to (1), $A(u(t))$ is a monotonic function in t. As an immediate consequence, nonconstant t-periodic solutions to (1) do not exist. Because of the gradient structure, one might hope to employ a Morse-theoretic approach to classifying the global solutions to (1).

1.2. **Summary of past work.** If instead we were to consider $x \in \Omega \subset \mathbb{R}^n$ for some bounded Ω, then the boundary-value problem that results has been discussed extensively in the literature. [5] [6] [1] For instance, all bounded global solutions then tend to limits as $|t| \to \infty$, and these limits are equilibrium solutions. Without too much effort, one can show that a standard extension to Morse theory will then work in the generic case.

1.2.1. *Existence of global solutions.* However, for unbounded domains, there is less known. Of course, existence and uniqueness of solutions on short time intervals (on strips $(-t_0, t_0) \times \mathbb{R}^n$) has been shown using semigroup methods, and is entirely standard. [11] However, there are obstructions to global existence. Aside from the typical loss of regularity due to solving the backwards heat equation, there is also a blow-up phenomenon which can spoil existence in the forward-time solution to (1). [4] [12] Global solutions to (1) are rather rare. Most works which describe blow-up make the assumption that the solution is positive. Unfortunately, blow-up is much more difficult to characterize in the general situation, and understanding exactly what kind of initial conditions are responsible for blow-up in the Cauchy problem for (1) is an important part of my future work.

1.2.2. *Finite energy.* Most of my progress towards a Morse theory for (1) has been made under the assumption of a finite energy condition on solutions. We define an *energy* functional:

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial t}\right|^2 + |\Delta u + P(u)|^2 \, dx\, dt. \tag{3}$$

It is easy to see that if a solution tends to limits as $|t| \to \infty$ each of which have finite action, then the energy functional measures the difference between the action of these limits. If we restrict our attention to the equilibrium solutions for which the action functional A is finite, then I have shown that solutions to (1) have finite energy if and only if they connect pairs of equilibria. [8] Stated another way, finite energy solutions are exactly the heteroclinic orbits of (1). This fact is reminiscent of A. Floer's work on exact symplectomorphisms. [3]

My current work (for the Ph.D. thesis) centers around proving the finite- dimensionality results of Floer for (1). Namely, that the manifold of heteroclinic orbits which connect a given pair of equilibria is finite-dimensional, and that its dimension can be used to assign a relative Morse index to each equilibrium. More precisely, suppose x, y is a pair of finite-action equilibria, and $W(x, y)$ is the manifold of solutions connecting x to y. We look for an integer-valued function I on the equilibria, such that

$$\dim W(x, y) = I(y) - I(x).$$

Evidently, this function I will be defined only up to an additive constant. With this relative index in hand, we can construct an analog to the Morse-Smale-Witten complex (called the Floer complex), and use it to compute homology groups for the space of all finite energy solutions.

1.2.3. *Traveling wave techniques.* If the coefficients a_i are constant, then there is a translational symmetry present in the equation. Such a symmetry allows one to look for solutions of the form $u(t, x) = U(x - ct)$ for some c. (In higher dimensions, there is also a rotational symmetry, which gives rise to spiral waves.) Such a U can be found by solving an elliptic equation. Much of the work involved in examining traveling waves is in understanding how the wave speed parameter c is determined. Additionally, one can think of a traveling wave solution as stationary within a moving frame, and thereby look at the stability of the wave. [10] Unfortunately, when the translational symmetry is broken, the resulting *nonlinear* scattering problem cannot be examined using these traveling wave techniques. Addressing this important issue is crucial to understanding global solutions to semilinear parabolic equations.

1.2.4. *Equilibrium solutions.* Since the finite energy solutions to (1) connect finite-action equilibria, it is also important to understand the collection of equilibrium solutions, which are global solutions to the elliptic problem

$$0 = \Delta u + \sum_{i=0}^{N} a_i u^i \tag{4}$$

on all of $\mathbb{R}^n$. Like (1), there are obstructions to global existence in (4). [9] Indeed, there are fairly few global solutions to (4). I have recently solved this problem in one dimension ($n = 1$) under asymptotic decay conditions for the a_i. The solution reveals delicate bifurcation behavior in the number of equilbria as the coefficients

a_i are varied. Further, the asymptotic behavior is such that all global solutions to (4) have finite action. [7]

2. Future work

2.1. Relaxation of the finite energy condition.

While the finite energy condition is a natural way to simplify the problem of classifying solution behavior in (1), it eliminates much of the interesting behavior. Indeed, it is easy to see that a traveling wave solution will not have finite energy. It is therefore imperative to relax the finite energy condition, and to consider all equilibria, not just those of finite action.

New techniques will need to be developed to handle this more general situation. In particular, the results described above for traveling waves do not work when the coefficients a_i vary in space, as this breaks the symmetry. Additionally, the finite energy methods rely on the finite energy condition in an essential way. However, it is not unreasonable to suppose that the finite energy and traveling wave methods can be combined using asymptotic matching.

A good example of this combination is the phenomenon of "launching" or "catching" a traveling wave by a finite action equilibrium. Ignoring regularity considerations for the moment, consider

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u^2(t,x) - H(x)u(t,x), \tag{5}$$

where $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$. Evidently the zero function is a finite action equilibrium in this case, and the function H is an infinite action equilibrium. There is a 1-parameter family of solutions which connect H to the zero function, and they have the appearance of bump functions, whose rightmost boundary is moving to the left. (This is easy to see if we consider the Cauchy problem for (5), and consider finite energy solutions restricted to the half-plane $[0, \infty) \times \mathbb{R}$.) From the point of view of an asymptotic analysis, this family of solutions looks like traveling waves for large x, and like finite energy solutions for small x.

Motivated by this example, I propose the following conjecture: that bounded global solutions to (1) always tend to limits as $|t| \to \infty$, and these limits are equilibrium solutions. If this conjecture is true, then we can attempt to construct a new complex of solutions, as in the case of finite energy. This complex will, however, be *infinite*-dimensional since there is no hope of the connecting manifolds being finite-dimensional in general. (One only needs to consider the case of (5) to realize that there are infinitely many linearly independent families of connecting solutions.) As a result, it will not be possible to compute the homology of the space of all global solutions, at least not by using any standard technique.

2.2. Improvements to the equilibrium analysis.

It should be noted that the equilibrium analysis in [7] does not explicitly require a finite action condition for the equilibria. However, this is a natural consequence of the decay conditions chosen for the a_i. So to support the possibility of solutions with infinite energy, the equilibrium analysis needs to be generalized somewhat. If the a_i have well-defined limits as $|x| \to \infty$, then the analysis of [7] should generalize to that case reasonably well. (Du and Ma [2] have made some progress in this direction.) A key difference between this situation and the one with the decay condition of [7] is that there may be uncountably many solutions to (4).

2.3. **Impact of this problem.** My work will characterize the behavior of all heteroclinic orbits of (1). Understanding the heteroclinic orbits of a dynamical system povides a "high-level" state-transition diagram for its dynamics, which is important for forming qualitative models of complicated systems. As a result, this would allow the behavior of a reaction-diffusion system within a more complicated system to be replaced by a qualitative (perhaps finite) state model.

From a mathematical point of view, the nonlinearity in (1) is a good choice for theoretical study. It is somewhat more specific than a general function of u and x, which allows sharper results to be obtained, especially since it ensures that (1) describes a gradient flow. Solution of this problem provides a framework for generalizing Morse theory into the realm of partial differential equations. Also, many nonlinearities that are of interest in applications are analytic in u, so they can be well-approximated by (1).

The work will also illuminate some fairly difficult topics in nonlinear analysis, by completely characterizing the solutions to a parabolic equation on an unbounded domain. In particular, it will highlight the interaction between traveling wave states and equilibrium states. Current works on traveling waves typically avoid this interaction, assuming that waves are a product of an unbroken symmetry and are essentially eternal. This is unrealistic: a flame traveling along a match clearly starts at one end! My work will examine the details of the ignition event, and how it effects the long-term behavior of the resulting waves.

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