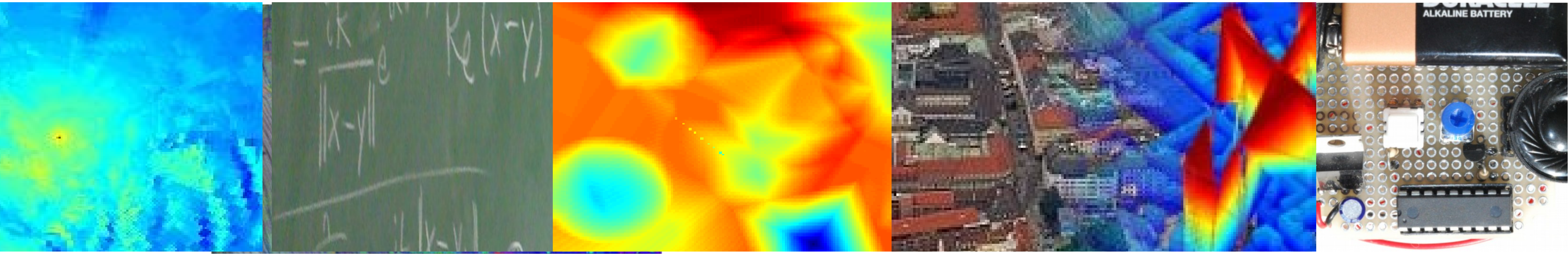
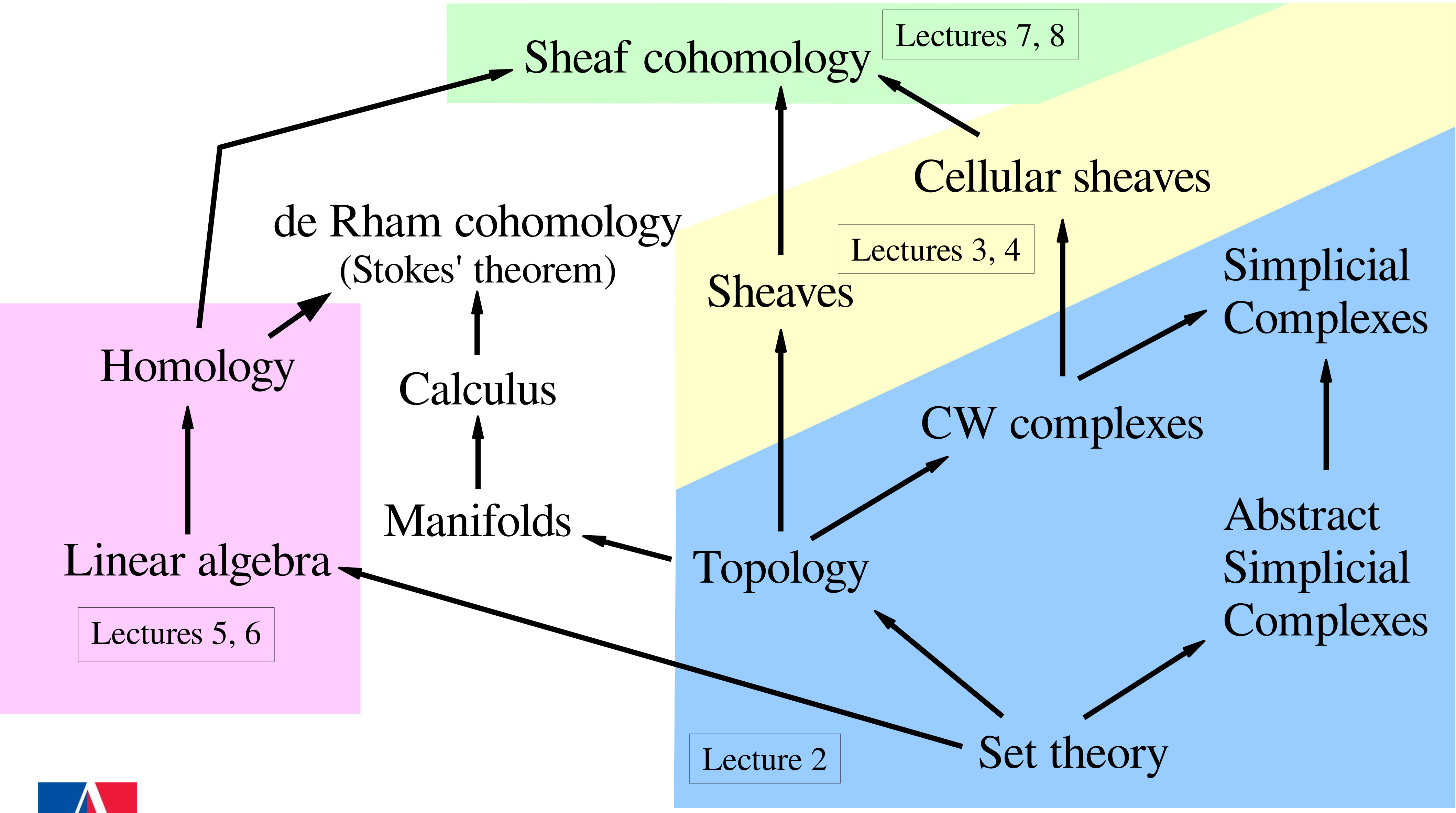
Sheaf Cohomology and its Interpretation



Michael Robinson

DARPA SIMPLEX Program

Mathematical dependency tree



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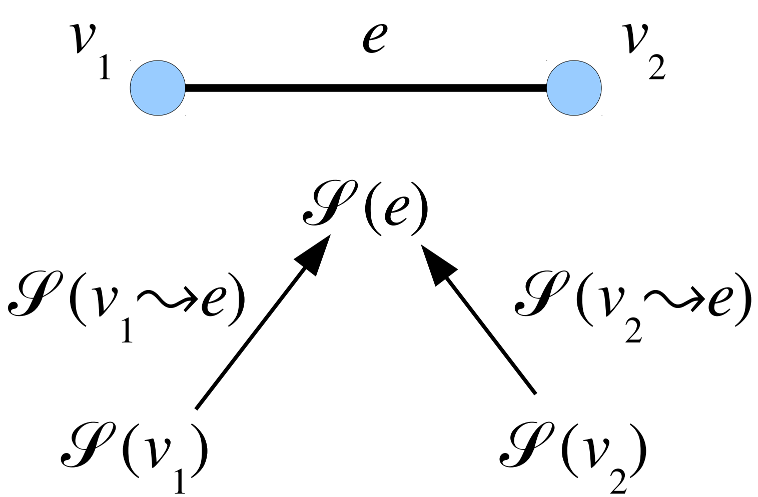
Session objectives

- What do global features of fused data look like?
- What new do the other sheaf invariants tell you?

Global sections, revisited

- The space of global sections alone is insufficient to detect redundancy or possible faults, but another invariant works
- It's based on the idea that we can rewrite the basic condition(s) for a global section s of a sheaf $\mathcal{S}$



$$\mathcal{S}(v_1 \rightsquigarrow e)\, s(v_1) = \mathcal{S}(v_2 \rightsquigarrow e)\, s(v_2)$$

$$+\,\mathcal{S}(v_1 \rightsquigarrow e)\, s(v_1) - \mathcal{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

$$-\,\mathcal{S}(v_1 \rightsquigarrow e)\, s(v_1) + \mathcal{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

($\mathcal{S}(a \rightsquigarrow b)$ is the restriction map connecting cell a to a cell b in a sheaf $\mathcal{S}$)

Global sections, revisited

- The space of global sections alone is insufficient to detect redundancy or possible faults, but another invariant works
- It's based on the idea that we can rewrite the basic condition(s) for a global section s of a sheaf $\mathscr{S}$

$$\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) = \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2)$$

$$\left(+\mathscr{S}(v_1 \rightsquigarrow e) \quad -\mathscr{S}(v_2 \rightsquigarrow e)\right)\begin{pmatrix} s(v_1) \\ s(v_2) \end{pmatrix} = 0$$

$$+\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) - \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

$$-\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) + \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

($\mathscr{S}(a \rightsquigarrow b)$ is the restriction map connecting cell a to a cell b in a sheaf $\mathscr{S}$)

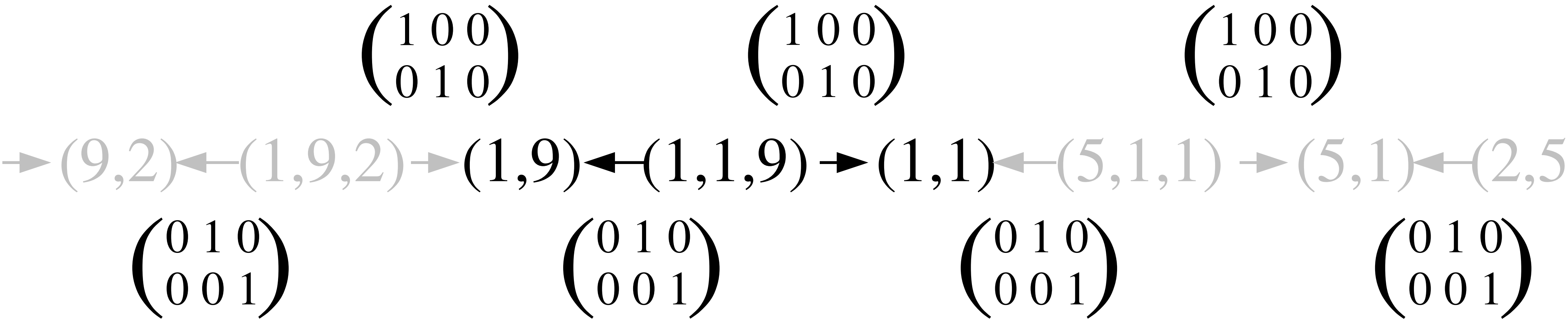
Recall: A queue as a sheaf

- Contents of the shift register at each timestep
- $N = 3$ shown

$$\rightarrow \mathbb{R}^2 \xleftarrow{\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} \mathbb{R}^2 \xleftarrow{\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} \mathbb{R}^2 \xleftarrow{\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} \mathbb{R}^2 \xleftarrow{\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}} \mathbb{R}$$

Recall: A single timestep

- Contents of the shift register at each timestep
- $N = 3$ shown



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Rewriting using matrices

- Same section, but the condition for verifying that it's a section is now written linear algebraically

$$(1,9,2) \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} (1,9) \xleftarrow[\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}]{} (1,1,9) \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} (1,1) \xleftarrow[\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}]{} (5,1,1)$$

$$\begin{pmatrix} 1&0&0 & 0&-1&0 & 0&0&0 \\ 0&1&0 & 0&0&-1 & 0&0&0 \\ 0&0&0 & 1&0&0 & 0&-1&0 \\ 0&0&0 & 0&1&0 & 0&0&-1 \end{pmatrix} \begin{pmatrix} 1\\9\\2\\1\\1\\9\\5\\1\\1 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}$$

The *cochain complex*

- Motivation: Sections being in the kernel of matrix suggests a higher dimensional construction exists!
- Goal: build the *cochain complex* for a sheaf $\mathscr{S}$

$$; \mathscr{S}) \xrightarrow{d^{k-1}} C^k(X; \mathscr{S}) \xrightarrow{d^k} C^{k+1}(X; \mathscr{S}) \xrightarrow{d^{k+1}} C^{k+2}(X;$$

- From this, *sheaf cohomology* will be defined as

$$H^k(X; \mathscr{S}) = \ker d^k \,/\, \text{image } d^{k-1}$$

much the same as homology (but the chain complex goes up in dimension instead of down)

Notational interlude

Homology

$C_k(X)$: chain space

∂_k : boundary map

$H_k(X)$: homology space

Dimensions go down in chain complex

Cohomology

$C^k(X)$: cochain space

d^k : coboundary map

$H^k(X)$: cohomology space

Dimensions go up in cochain complex

Generalizing up in dimension

- Global sections lie in the kernel of a particular matrix
- We gather the domain and range from stalks over vertices and edges... These are the *cochain spaces*

$$C^k(X; \mathscr{S}) = \bigoplus_{a \text{ is a } k\text{-simplex}} \mathscr{S}(a)$$

- An element of $C^k(X; \mathscr{S})$ is called a *cochain*, and specifies a datum from the stalk at each k-simplex

(The *direct sum* operator $\bigoplus$ forms a new vector space by concatenating the bases of its operands)

The cochain complex

- The *coboundary map* $d^k : C^k(X; \mathscr{S}) \to C^{k+1}(X; \mathscr{S})$ is given by the block matrix

$$d^k = \begin{pmatrix} & & \vdots & & \\ - - - - & [b_i : a_j] & \mathscr{S}(a_j \rightsquigarrow b_i) & - - - - \\ & & \vdots & & \end{pmatrix}$$

Row i

Column j

0, +1, or -1 depending on the relative orientation of a_j and b_i

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The cochain complex

- We've obtained the *cochain complex*

$$\cdots ; \mathscr{S}) \xrightarrow{d^{k-1}} C^k(X; \mathscr{S}) \xrightarrow{d^k} C^{k+1}(X; \mathscr{S}) \xrightarrow{d^{k+1}} C^{k+2}(X; \cdots$$

- *Cohomology* is defined as

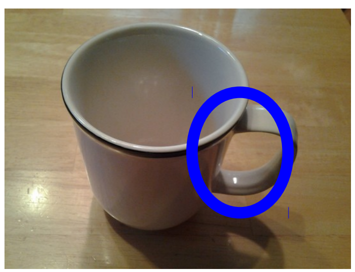
$$H^k(X; \mathscr{S}) = \ker d^k \,/\, \text{image } d^{k-1}$$

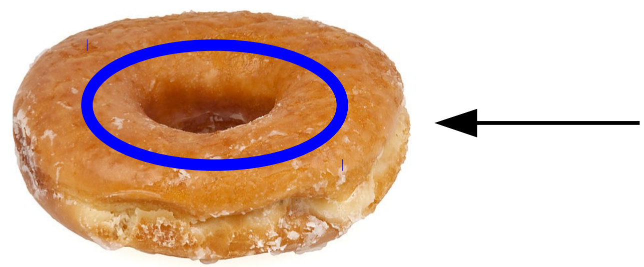
All the cochains that are consistent in dimension k ...

… that weren't already present in dimension k - 1

Cohomology facts

- $H^0(X; \mathscr{S})$ is the space of global sections of $\mathscr{S}$
- $H^1(X; \mathscr{S})$ usually has to do with oriented, non-collapsible **data** loops





Nontrivial $H^1(X; \mathbb{Z})$

- $H^k(X; \mathscr{S})$ is a functor: sheaf morphisms induce linear maps between cohomology spaces

Cohomology versus homology

Homologies of different chain complexes:

- Chain complex: simplices and their boundaries

$$\longrightarrow C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}}$$

- Transposing the boundary maps yields the *cochain complex*: functions on simplices

$$\xleftarrow{\partial^T_{k+2}} C_{k+1}(X) \xleftarrow{\partial^T_{k+1}} C_k(X) \xleftarrow{\partial^T_k} C_{k-1}(X) \xleftarrow{\partial^T_{k-1}}$$

- With $\mathbb{R}$ linear algebra, homology* of both of these carry identical information for a wide class of spaces

* we call the homology of a cochain complex *cohomology*

Cohomology versus homology

Homologies of different chain complexes:

- Transposing the boundary maps yields the *cochain complex*: functions on simplices

$$\xleftarrow{\partial^T_{k+2}} C_{k+1}(X) \xleftarrow{\partial^T_{k+1}} C_k(X) \xleftarrow{\partial^T_{k}} C_{k-1}(X) \xleftarrow{\partial^T_{k-1}}$$

The *co*boundary maps work like discrete derivatives and compute differences between functions on higher dimensional simplices

Sheaf cohomology versus homology

Homologies of different chain complexes:

- Transposing the boundary maps yields the *cochain complex*: functions on simplices

$$\xleftarrow{\partial_{k+2}^T} C_{k+1}(X) \xleftarrow{\partial_{k+1}^T} C_k(X) \xleftarrow{\partial_k^T} C_{k-1}(X) \xleftarrow{\partial_{k-1}^T}$$

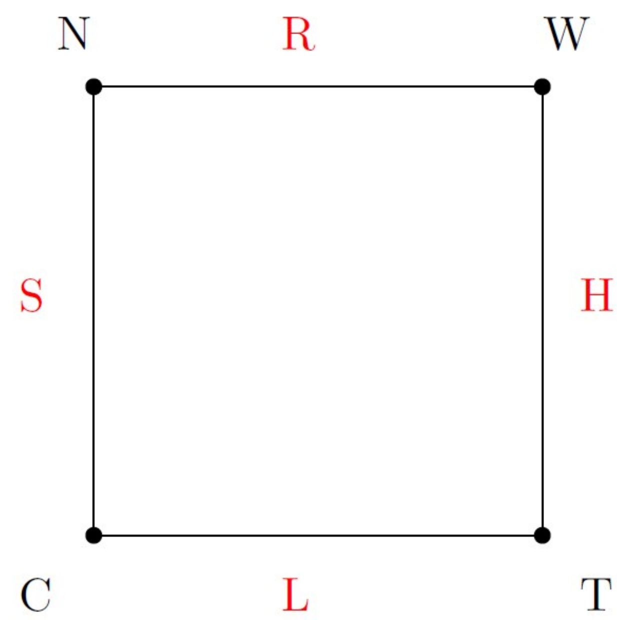
- *Sheaf cochain complex*: also functions on simplices, but they are generalized!

$$X; \mathcal{S}) \xleftarrow{d^{k+1}} C^{k+1}(X; \mathcal{S}) \xleftarrow{d^k} C^k(X; \mathcal{S}) \xleftarrow{d^{k-1}} C^{k-1}(X;$$

"Weather Loop" a simple model

Sensors/ Questions	Rain? (R)	Humidity % (H)	Clouds? (L)	Sun? (S)
News (N)	X			X
Weather Website (W)	X	X		
Rooftop Camera (C)			X	X
Twitter (T)		X	X	

Make simplicial complex



Question: Can misleading globalized information be detected?

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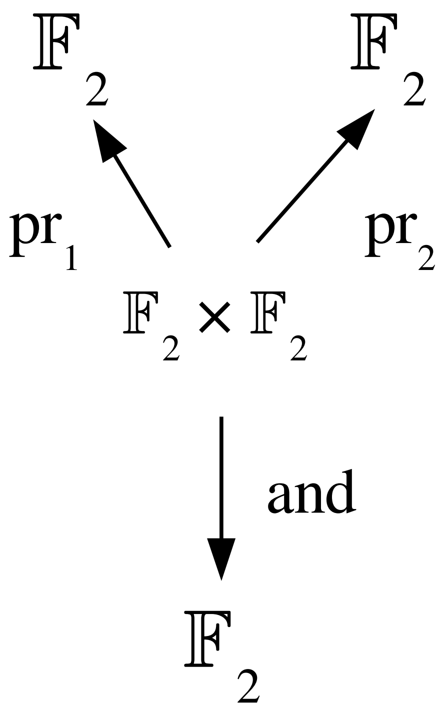
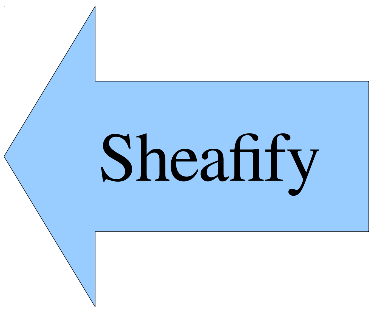
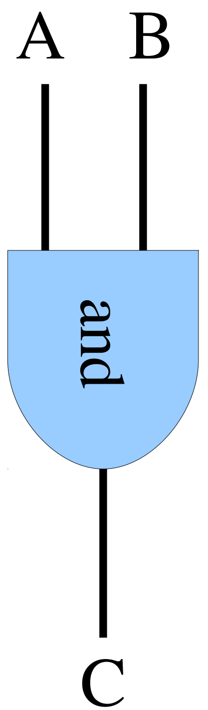
Switching sheaves

- It's possible to construct a sheaf that represents the truth table of a logic circuit
- Each vertex is a logic gate, each edge is a wire



Quiescent* logic sheaf





Logic circuit

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

*Quiescent = steady state, pr_n = projection onto *n*th component

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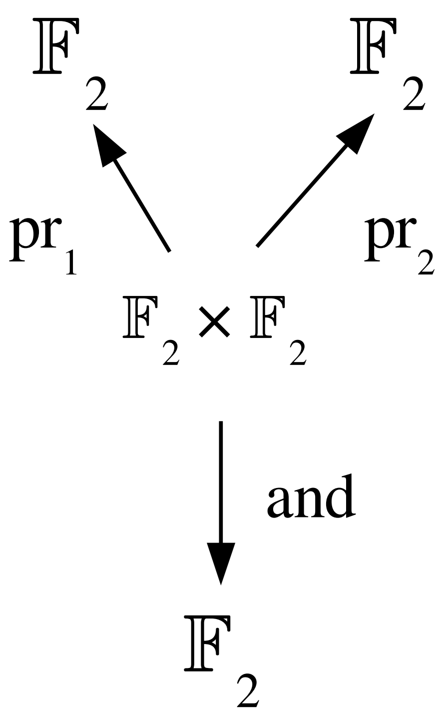
Switching sheaves

- Vectorify **everything** about a quiescent logic sheaf, and you obtain a *switching sheaf*



Quiescent* logic sheaf

$\mathbb{F}_2[\mathbb{F}_2 \times \mathbb{F}_2]$

=

A vector space whose basis is the set of ordered pairs

=

$\mathbb{F}_2^{\,2} \otimes \mathbb{F}_2^{\,2}$

$\otimes$ = Tensor product

*Quiescent = steady state, pr_n = projection onto *n*th component

Switching sheaves

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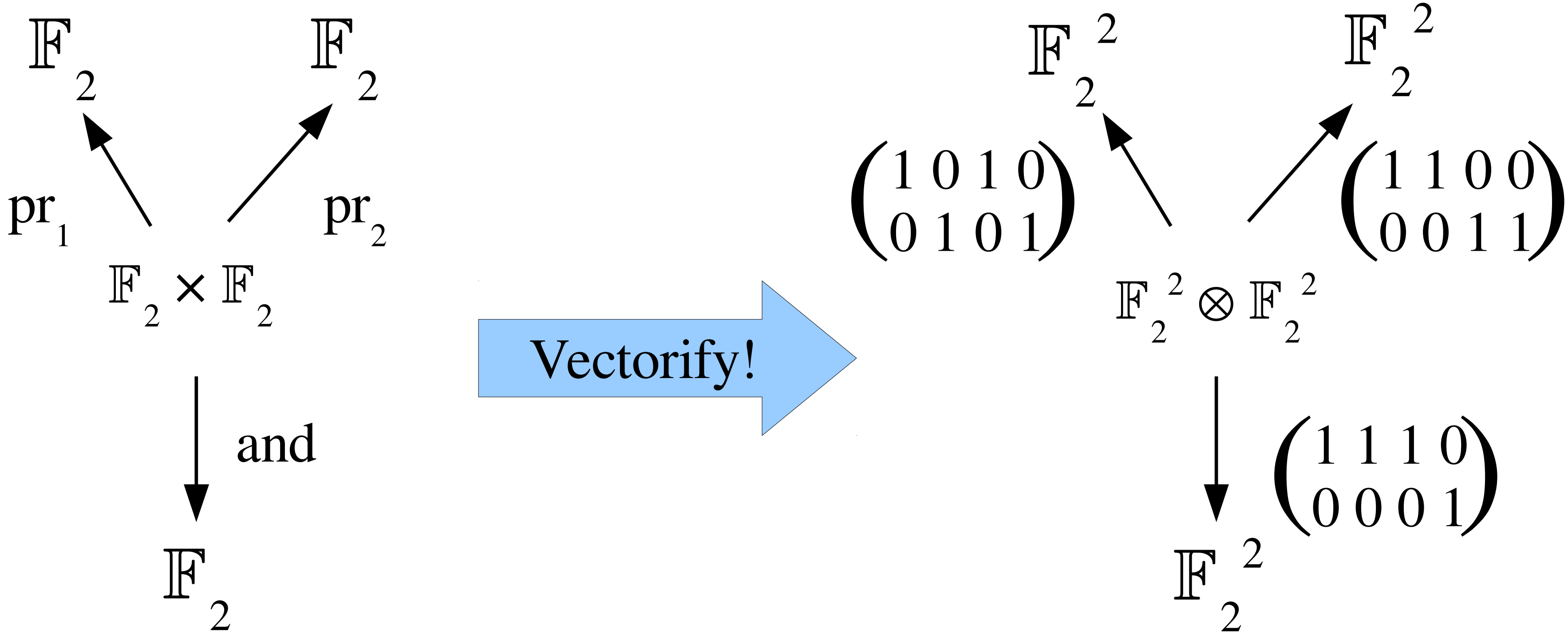


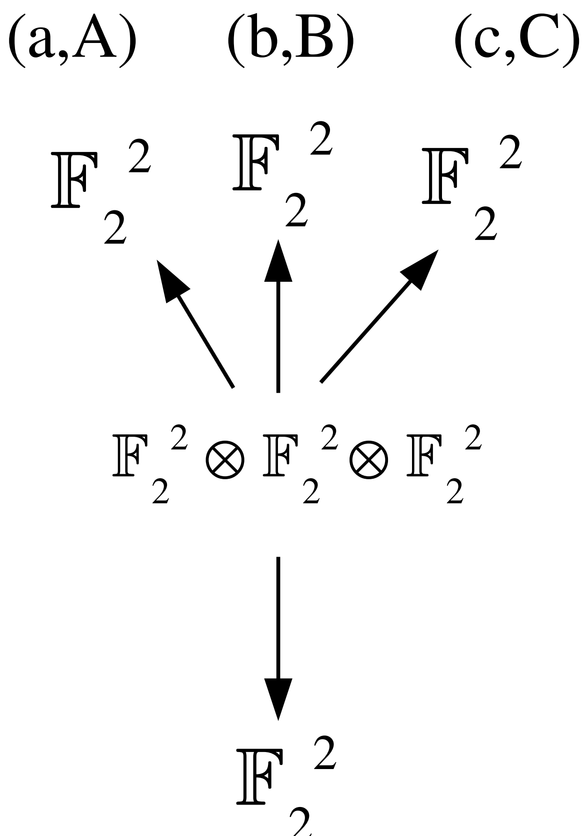
Quiescent* logic sheaf

Switching sheaf

*Quiescent = steady state, pr_n = projection onto *n*th component

Global sections of switching sheaves

- In the case of a 3 input gate, the global sections are spanned by **all simultaneous combinations** of inputs



$a \otimes b \otimes c$

$a \otimes b \otimes C$

$a \otimes B \otimes c$

$a \otimes B \otimes C$

$A \otimes b \otimes c$

$A \otimes b \otimes C$

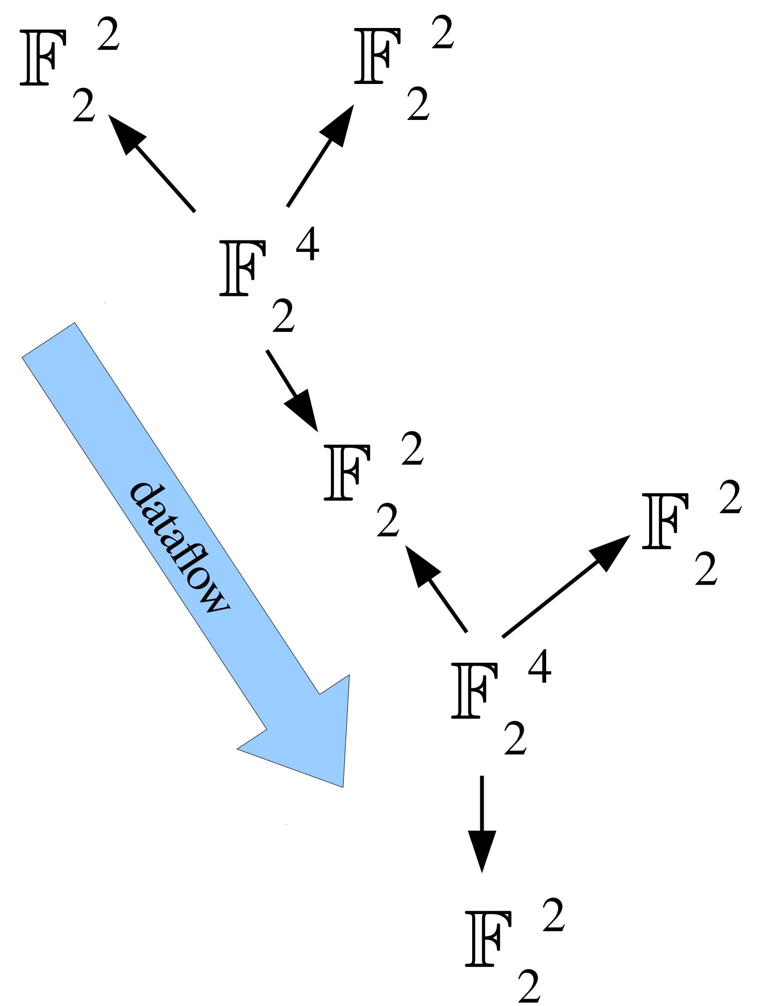
$A \otimes B \otimes c$

$A \otimes B \otimes C$

$2^8 = 256$ sections in total

Global sections of switching sheaves

- When we instead consider a logically equivalent circuit, the situation changes
- Global sections consist of simultaneous inputs to each gate, but consistency is checked via tensor contractions
- There is an inherent model of uncertainty



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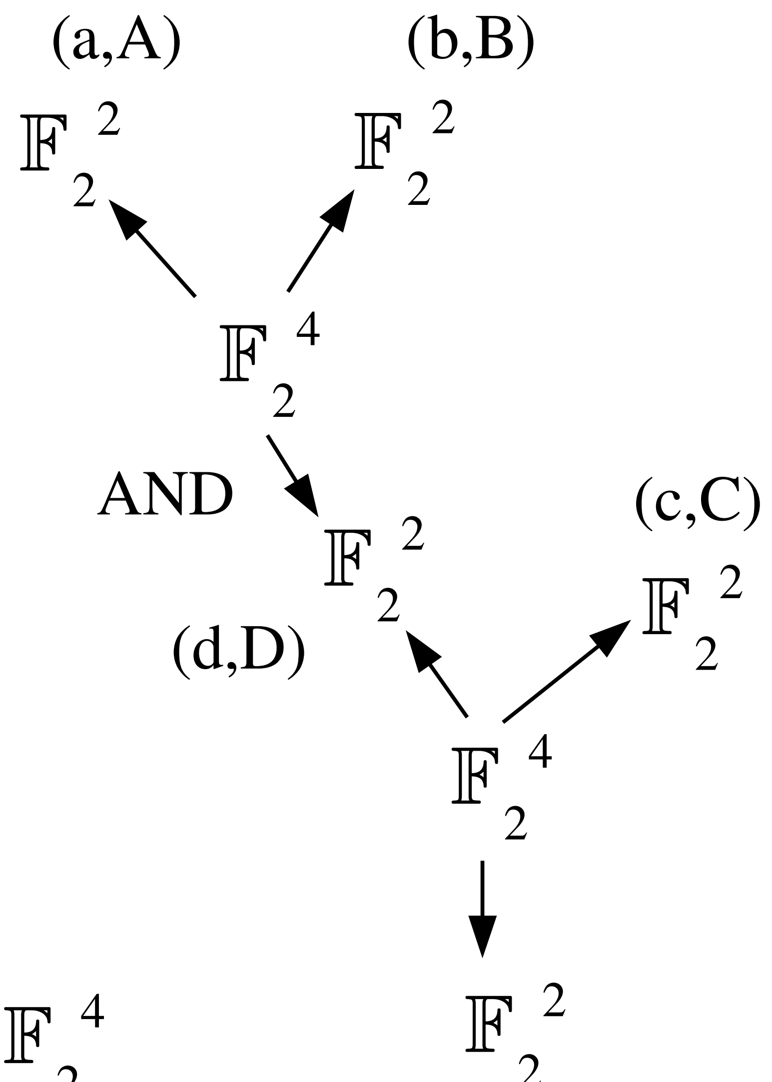
Global sections of switching sheaves

- The space of global sections is now 6 dimensional – some sections were lost!

$$a \otimes b + c \otimes d$$
$$a \otimes b + C \otimes d$$
$$a \otimes B + c \otimes d$$
$$A \otimes b + c \otimes d$$
$$A \otimes B + c \otimes D$$
$$A \otimes B + C \otimes D$$



Recall that the space of global sections is a subspace of $\mathbb{F}_2^4 \oplus \mathbb{F}_2^4$

Global sections of switching sheaves

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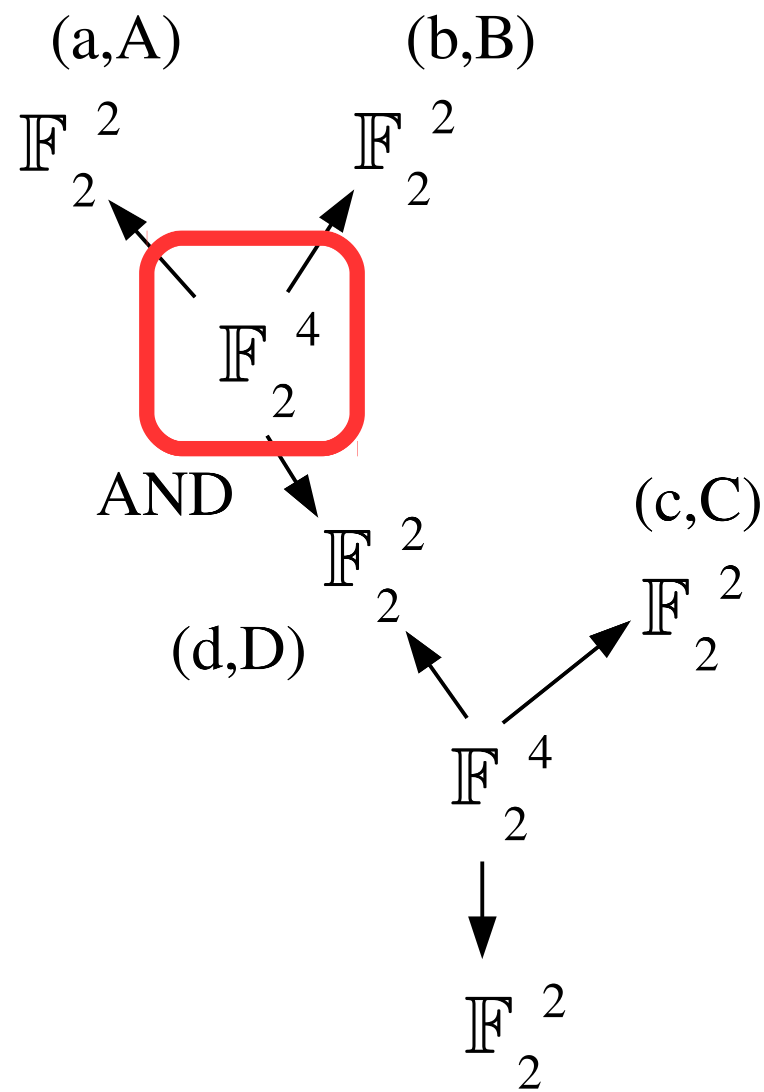
$$a \otimes B + c \otimes d$$
$$A \otimes b + c \otimes d$$
$$A \otimes B + c \otimes D$$
$$A \otimes B + C \otimes D$$

- All local sections on the upstream gate are represented



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Global sections of switching sheaves

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$$a \otimes b + c \otimes d$$
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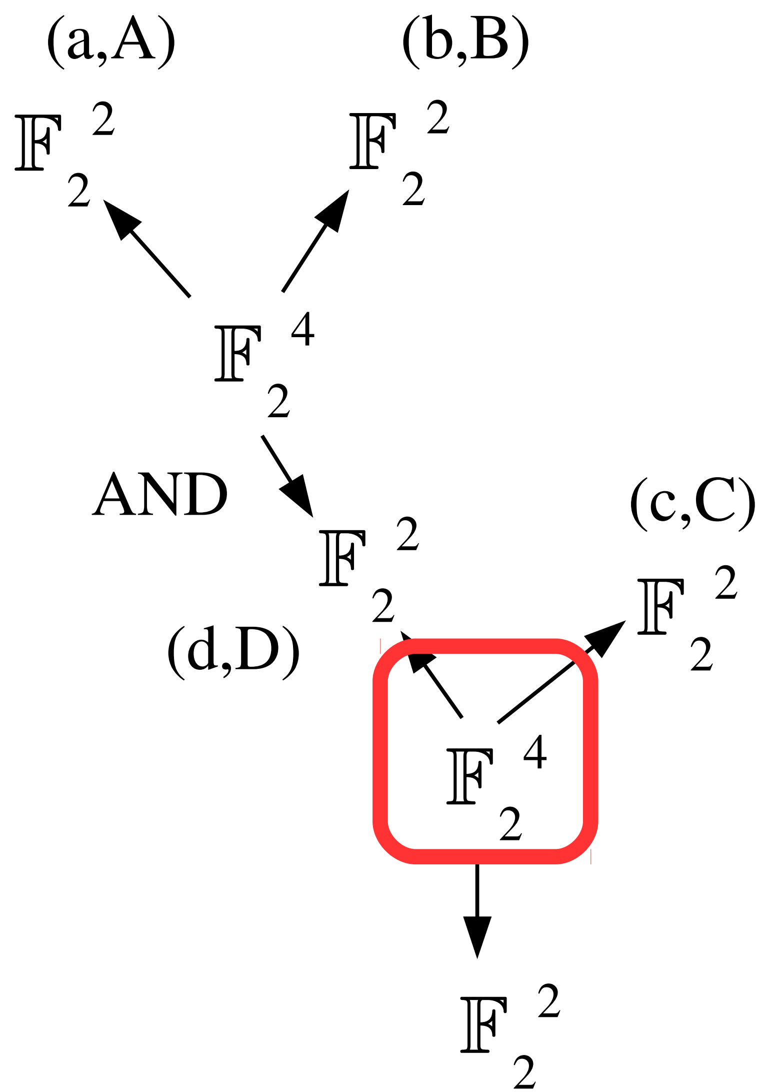
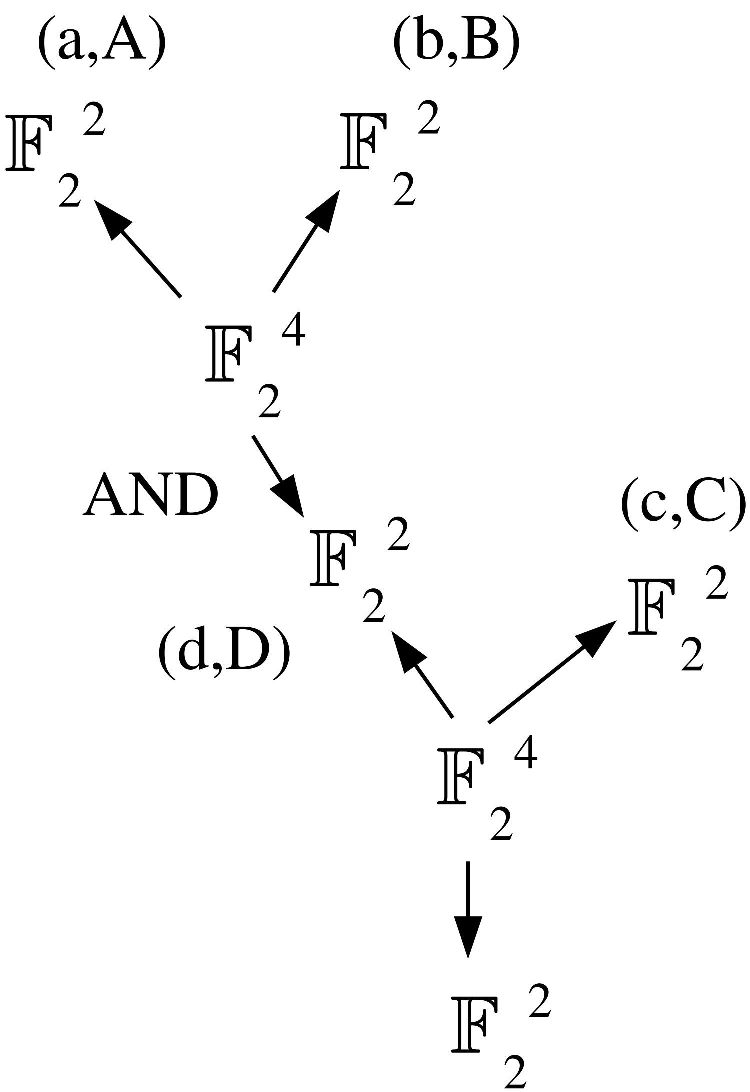
$$a \otimes B + c \otimes d$$
$$A \otimes b + c \otimes d$$
$$A \otimes B + c \otimes D$$
$$A \otimes B + C \otimes D$$

- All local sections supported on the downstream gate are there too



Global sections of switching sheaves

No quiescent logic states were actually lost, but the sections of this sheaf represent sets of simultaneous data at each gate that might be **in transition**!



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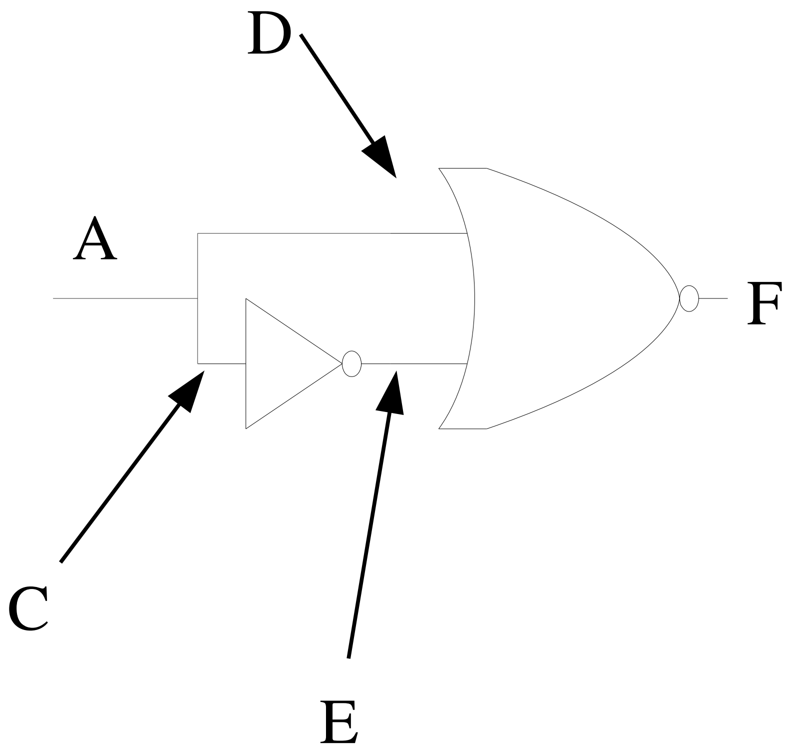
Higher cohomology spaces

- Switching sheaves are written over 1-dimensional spaces, so they could have nontrivial 1-cohomology
- Nontrivial 1-cohomology classes consist of **directed loops that store data**
- Since we just found that logic value transitions are permitted, this means that 1-cohomology can detect glitches

Glitch generator: cohomology



$H^0(X;\mathscr{F})$ is generated by

A+C+D ⊗ e
a+c+d ⊗ E

A+a+C+c+d ⊗ e+D ⊗ E

Indication that there's a race condition possible → $H^1(X;\mathscr{F})\cong\mathbb{Z}_2$

Hazard transition state

H^1 detects the race condition

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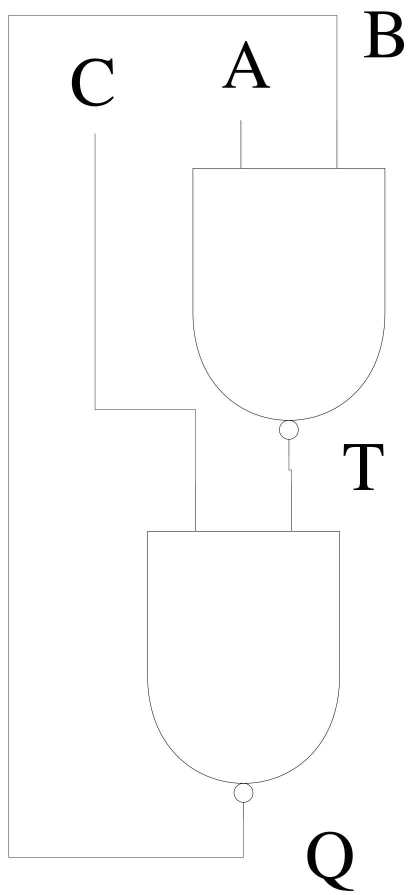
Example: flip-flop

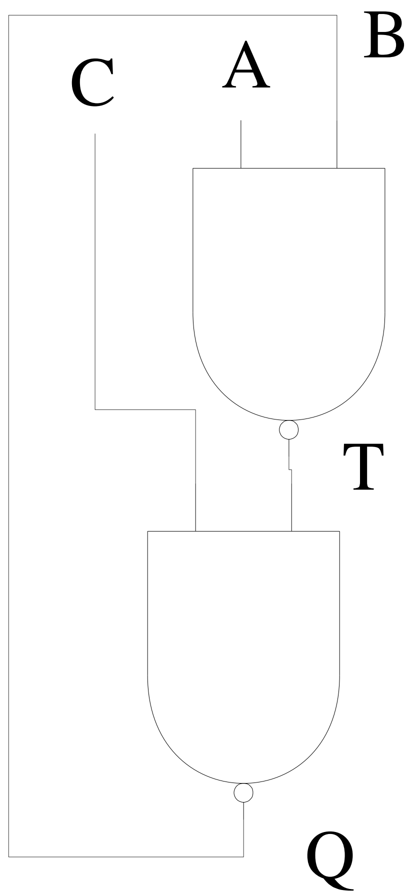


C	A	B	T	Q	
0	0	0	1	1	
0	0	1	1	1	Hazard!
0	1	0	1	1	
0	1	1	0	1	Set
1	0	0	1	0	Reset
1	0	1	1	0	
1	1	0	1	0	Hold
1	1	1	0	1	Hold

Transition out of the hazard state to the hold state causes a race condition

This is what traditional analysis gives... 5 possible states

Flip-flop cohomology



$H^1(X;F) \cong \mathbb{Z}_2$ ← Race condition detected!

$H^0(X;F) \cong \mathbb{Z}_2^7$

Generated by:

$a \otimes B \otimes c$

$A \otimes B \otimes c$

$a \otimes b \otimes C$

$A \otimes b \otimes C$

$A \otimes B \otimes C$

$a \otimes b \otimes c + a \otimes B \otimes C$

$a \otimes b \otimes c + A \otimes b \otimes c$

These states describe the possible transitions out of the hazard state – something that takes a bit more trouble to obtain traditionally

States from the truth table

Bonus: Cosheaf homology

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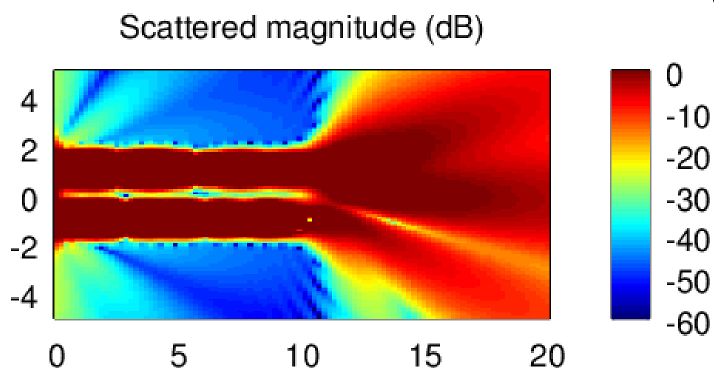
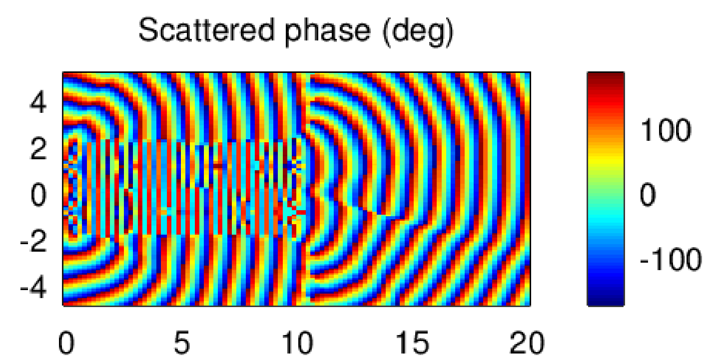
Cosheaf homology

- The globality of cosheaf sections concentrates in top dimension, which may vary over the base space
  - No particular degree of cosheaf homology holds global sections if the model varies in dimension
- But what **is** clear is that numerical instabilities can arise if certain nontrivial homology classes exist
  - These can obscure actual solutions, but can look "very real" resulting in confusion
  - There are many open questions…

Wave propagation as cosheaf

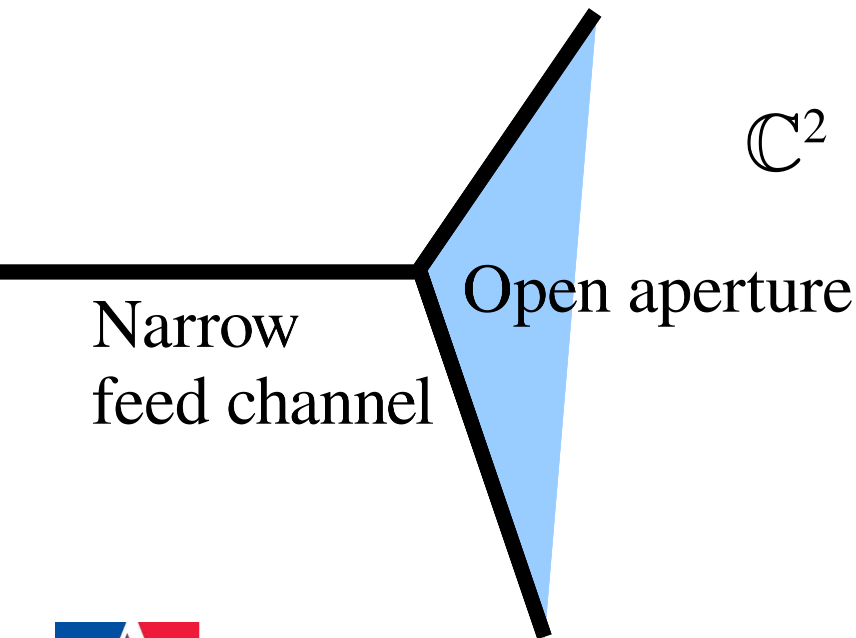
$\Delta u + k^2 u = 0$
with Dirichlet
boundary
conditions





Wave propagation as cosheaf

- Solving $\Delta u + k^2 u = 0$ (single frequency wave propagation) on a cell complex with Dirichlet boundary conditions



$$M([0,\infty),\mathbb{C}) \xrightarrow{ev_0} \mathbb{C}$$

$$\mathbb{C}^2 \xrightarrow{pr_1} \mathbb{C}$$

$$M((-\infty,0],\mathbb{C}) \xrightarrow{ev_0} \mathbb{C}$$

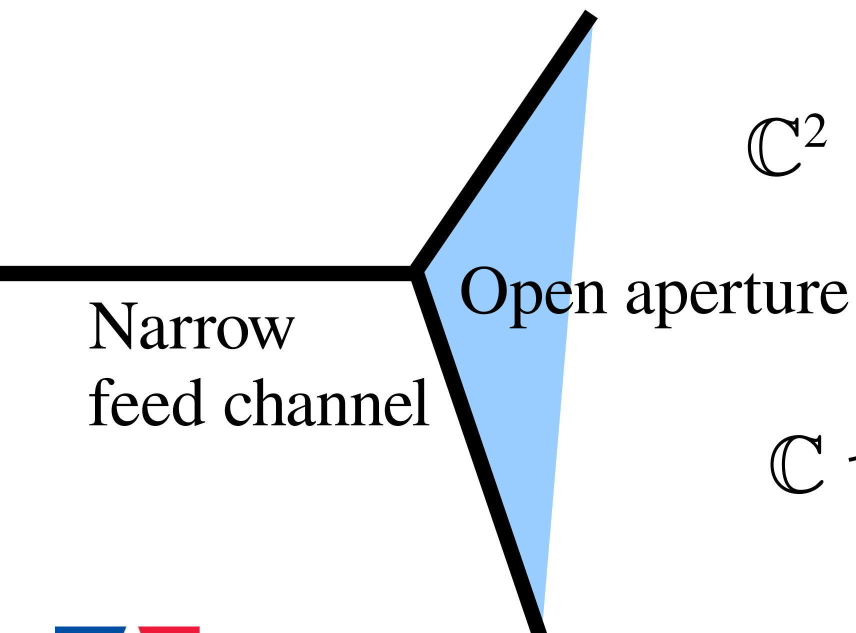
$M(S^1,\mathbb{C}) \to M([0,\infty),\mathbb{C})$ (an integral transform)

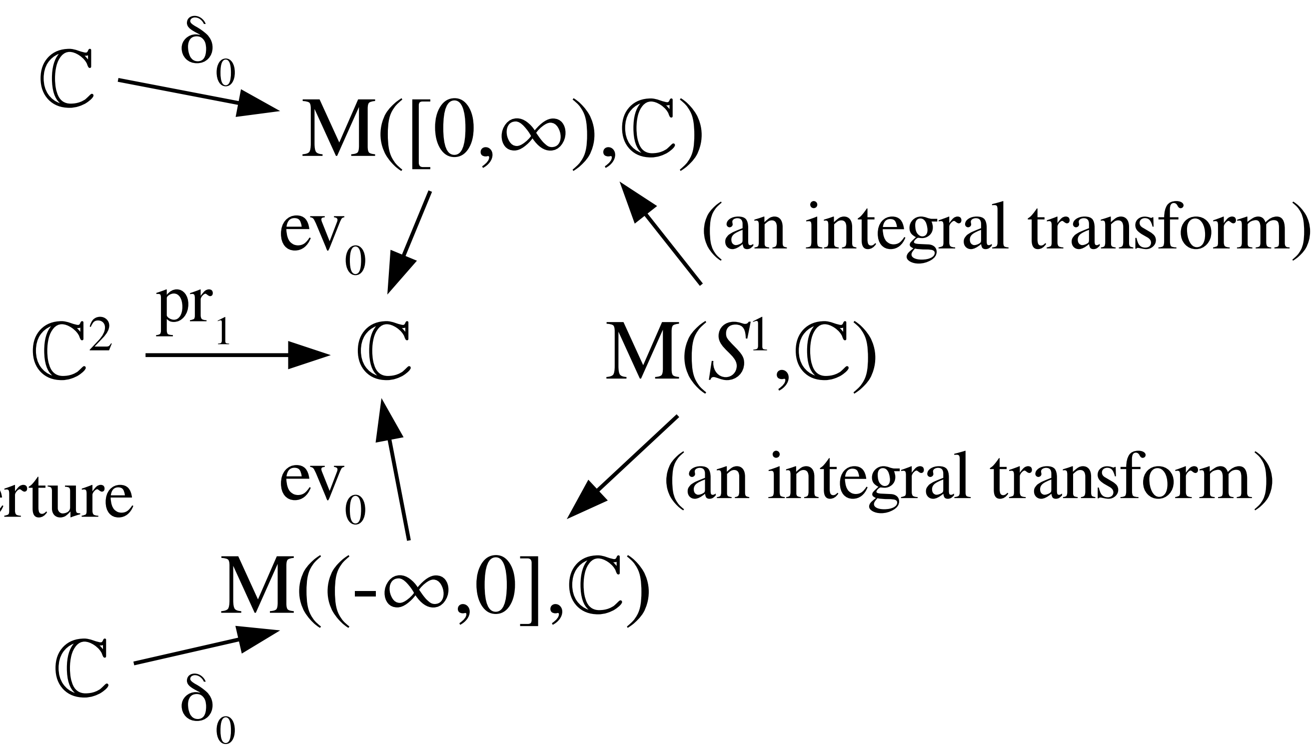
$M(S^1,\mathbb{C}) \to M((-\infty,0],\mathbb{C})$ (an integral transform)

$M(X,\mathbb{C})$ = Space of complex measures on X

Wave propagation as cosheaf

- Solving $\Delta u + k^2 u = 0$ (single frequency wave propagation) on a cell complex with **Dirichlet boundary conditions**





$M(X,\mathbb{C})$ = Space of complex measures on X

Wave propagation cosheaf homology

- The global sections indeed get spread across dimension
- Here's the chain complex:

Dimension 2 Dimension 1 Dimension 0

$$M(S^1,\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \rightarrow M((-\infty,0],\mathbb{C}) \oplus \mathbb{C}^2 \oplus M([0,\infty),\mathbb{C}) \longrightarrow \mathbb{C}$$

Global sections are parameterized by a subspace of these

Next up...

- Interactive session: Computing Homology and Cohomology
- Next and final lecture: How do we Deal with Noisy Data?

Further reading...

- Louis Billera, "Homology of Smooth Splines: Generic Triangulations and a Conjecture of Strang," *Trans. Amer. Math. Soc.*, Vol. 310, No. 1, Nov 1998.
- Justin Curry, "Sheaves, Cosheaves, and Applications" http://arxiv.org/abs/1303.3255
- Michael Robinson, "Inverse problems in geometric graphs using internal measurements," http://www.arxiv.org/abs/1008.2933
- Michael Robinson, "Asynchronous logic circuits and sheaf obstructions," *Electronic Notes in Theoretical Computer Science* (2012), pp. 159-177.
- Pierre Schapira, "Sheaf theory for partial differential equations," *Proc. Int. Congress Math.*, Kyoto, Japan, 1990.