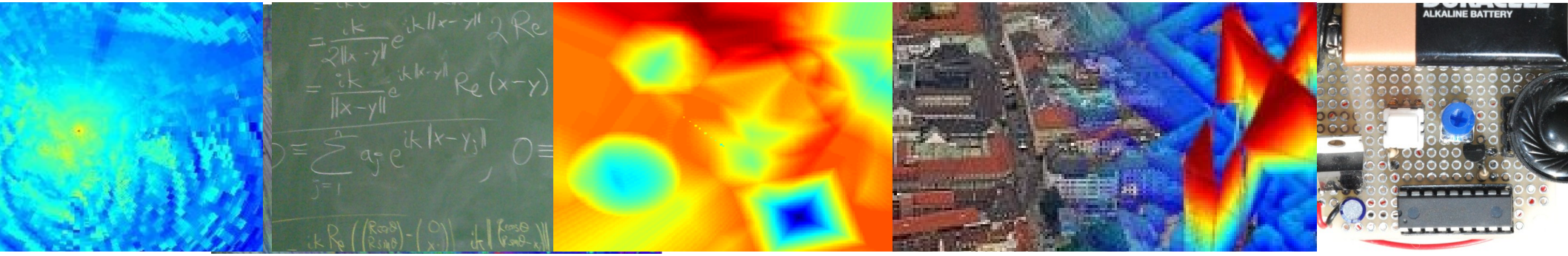
The Category of Binary Relations, Dowker complexes, Cosheaves, and Functoriality



Michael Robinson

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Acknowledgments

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  - Kris Ambrose, Steve Huntsman, Allyson O'Brien, Matvey Yutin
- Students:
  - Ken Ewing (MS Math Info & Sec)
- Funding:
  - Sergey Bratus (DARPA/I2O) SafeDocs

- Key reference:

  M. Robinson, "Cosheaf representations of relations and Dowker complexes."

  https://arxiv.org/abs/2005.12348

  (basically everything in these slides comes from this preprint)

Motivation: Consensus file formats

- What does it mean for files to comply with a format, especially if the format is an ambiguous, community-defined consensus?
- <u>Tactic</u>: Use a *binary relation,* recording which files are accepted as valid by which parsers
- <u>Hypothesis</u>: Anomalous files or parsers will manifest within the context of this relation

$$
\begin{array}{cc}
 & \text{files} \\
\text{parsers} & \begin{array}{c} A \\ B \\ C \\ D \end{array}
\begin{pmatrix}
10000011111100001111 \\
01100011100010000000 \\
00011000010011111111 \\
00000100001101111111
\end{pmatrix}
\end{array}
$$

1 = file parsed successfully
0 = problem parsing file

It's probably the case that there are many more files than parsers, but this isn't terribly crucial

Main ideas of the talk

- The category **Rel** of relations has a simplicial representation, the *Dowker complex*
  - The Dowker complex is a covariant functor
  - **Rel** applies whenever you have tabular data
  - Files accepted by parsers, extant key-value pairs, etc.
- The Dowker complex isn't a complete invariant, but it is when *weighted*
  - May enable statistical analysis of the topology within and between tables
- This leads to a faithful *cosheaf representation*
  - Topology is therefore a reliable description of a table
- The cosheaf carries both the Dowker complex and its *dual* (transpose of the relation)
  - One of the two Dowker complexes will usually be **much** bigger
  - You can use the cosheaf on the smaller complex without loss of structure

Binary relations in their category

- We should formalize the category **Rel** of relations
- Objects are triples (X,Y,R) where $R \subseteq X \times Y$
- These are best thought of as tables of booleans*

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$X = \{a, b, c, d, e\}$, $Y = \{1, 2, 3, 4, 5\}$

$R = \{(a,1), (a,2), (b,1), (b,3), (c,2), (c,3), (c,4), (c,5), (d,3), (d,4), (e,4), (e,5)\}$

*This generalizes neatly to entries with poset values

Morphisms in **Rel**

- A morphism in Rel consists of a pair of functions

  $\boldsymbol{f : X_1 \to X_2}$, $g : Y_1 \to Y_2$

  satisfying a compatibility condition

$$
\begin{array}{c}
a \\ b \\ c \\ d \\ e
\end{array}
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\xrightarrow{\;f\;}
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\begin{array}{c}
A \\ B \\ C
\end{array}
$$

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$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \xrightarrow{\;g\;} \begin{array}{c} \begin{array}{ccccc} g(1) & g(2) & g(3) & g(4) & g(5) \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

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 \\ a \\ b \\ c \\ d \\ e
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & \boxed{1 \quad 1 \quad 1 \quad 1} \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}
\xrightarrow{g,\ f}
\begin{array}{c}
\begin{array}{ccccc} g(1) & g(2) & g(3) & g(4) & g(5) \end{array} \\
\begin{pmatrix}
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\end{array}
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0 & 0 & \boxed{1} & \boxed{1} & 0 \\
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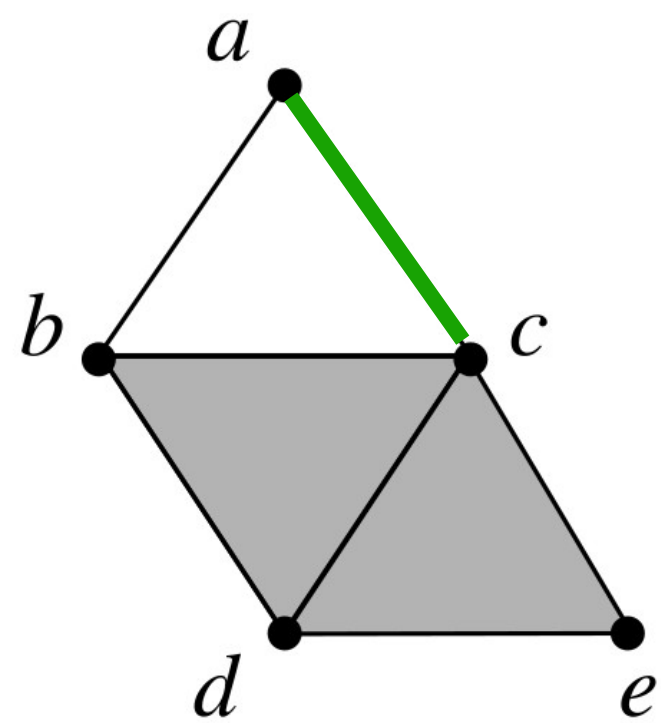
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
\begin{array}{c} \\ A \\ B \\ C \end{array}
$$

Classic rep'n: Dowker complex

- Each row specifies a vertex
- Each column specifies (at least one) simplex by selecting subsets of vertices

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
\end{array}
$$



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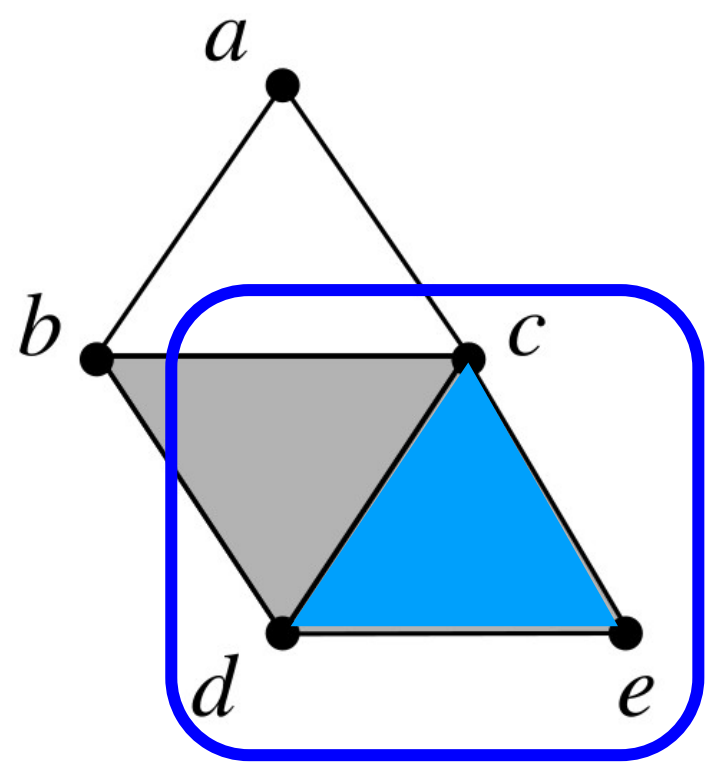
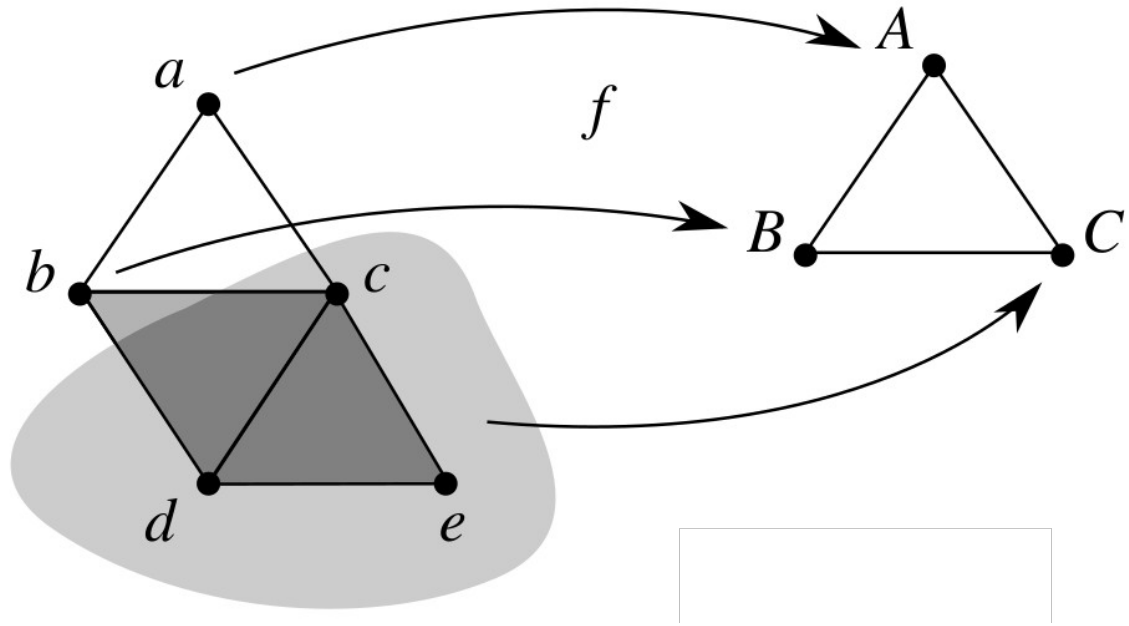
$$



The Dowker complex is functorial

<u>Theorem</u>: Functoriality for inclusions of relations (Chowdhury and Mémoli, *JACT*, 2018.)

<u>Theorem</u>: Also true at full generality!

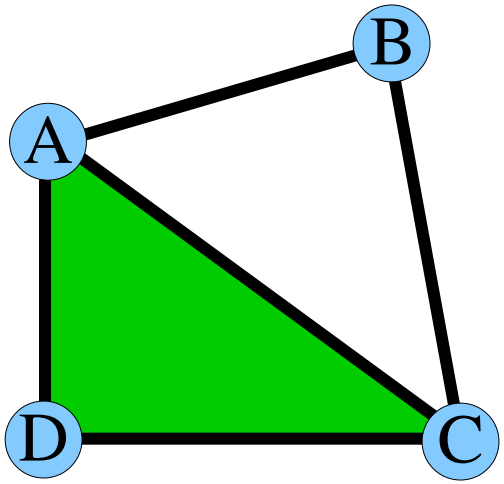


$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{array}{c} A \\ B \\ C \end{array}$$

Dowker is lossy

- Dowker ignores duplicate columns
- Here are several non-isomorphic relations inducing the same complex

$$\text{parsers}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{files} \\ \begin{pmatrix} 100000111111000011111 \\ 011000111000100000000 \\ 000110000100111111111 \\ 000001000011011111111 \end{pmatrix} \end{array}$$



$$\text{parsers}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{files} \\ \begin{pmatrix} 101001 \\ 110000 \\ 011101 \\ 001010 \end{pmatrix} \end{array}$$

$$\text{parsers}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{files} \\ \begin{pmatrix} 00110110 \\ 01100000 \\ 01011000 \\ 00010101 \end{pmatrix} \end{array}$$

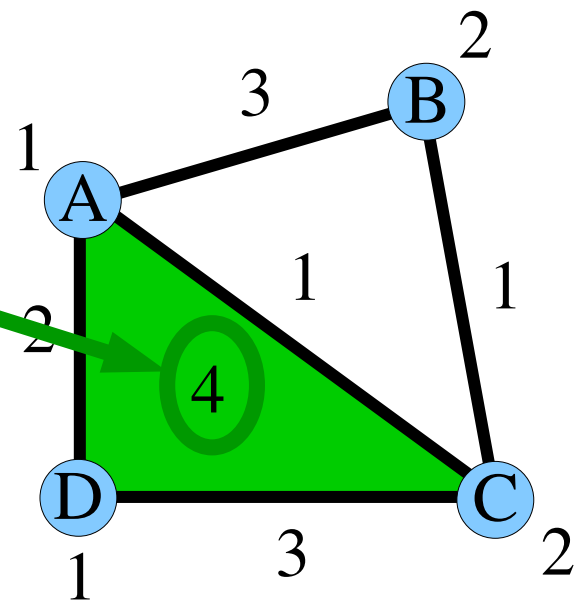
Weighted Dowker complex

- Weighting: Count how many times each simplex appears
- <u>Theorem</u>: The matrix is determined (up to isomorphism) by the Dowker complex with this weight function
- <u>Deeper theorem</u>: This can be enriched into a covariant functor

files

parsers

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 1000001111110000\boxed{1111} \\ 0110001110001000\boxed{0000} \\ 0001100001001111\boxed{1111} \\ 0000010000110111\boxed{1111} \end{pmatrix}$$

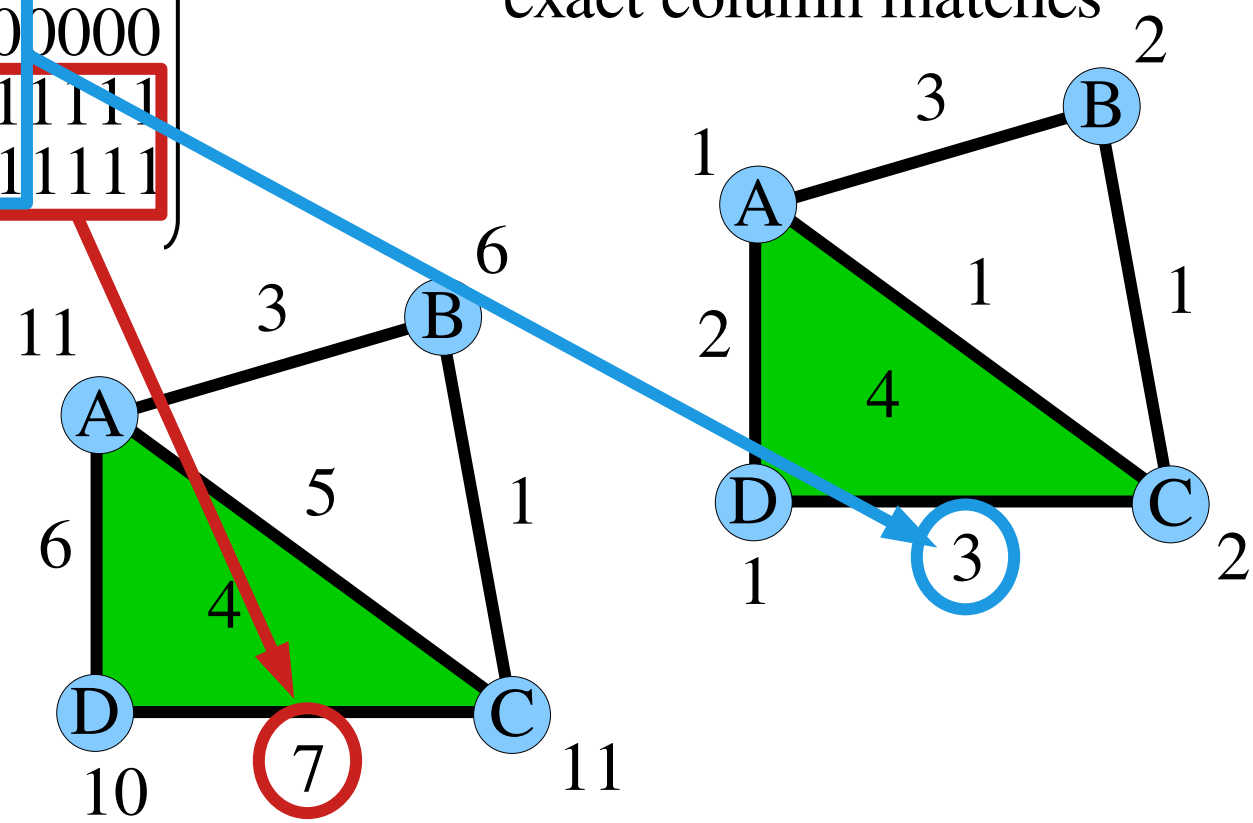


Dowker weight functions

- There are actually **two** weight functions that distinguish isomorphism classes

$$\text{parsers}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \overset{\text{files}}{\begin{pmatrix} 1000001111110001111 \\ 0110001110001000000 \\ 0001100001001111111 \\ 0000010000110111111 \end{pmatrix}}$$

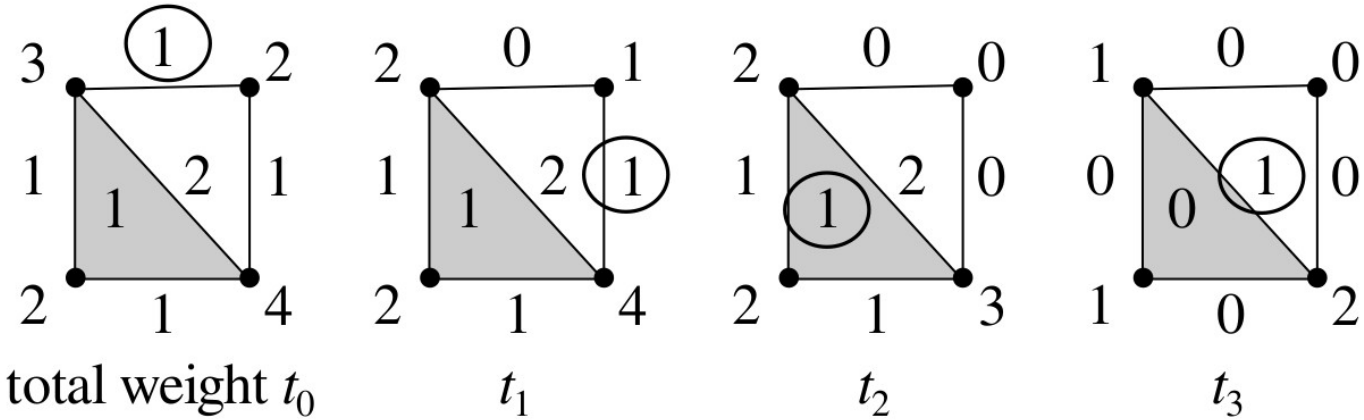
Differential weight:
For each simplex, count exact column matches



Total weight:
For each simplex, count columns matching the vertices present. Ignore the other vertices

Dowker weight functions

- <u>Theorem</u>: Both total and differential weight are complete isomorphism invariants for **Rel**.
- The proof is constructive and algorithmic



total weight t_0 t_1 t_2 t_3

… and so on!

$$r_0 = \begin{pmatrix} \end{pmatrix} \qquad r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 10 \\ 11 \\ 01 \\ 00 \end{pmatrix} \qquad r_3 = \begin{pmatrix} 101 \\ 110 \\ 011 \\ 001 \end{pmatrix}$$

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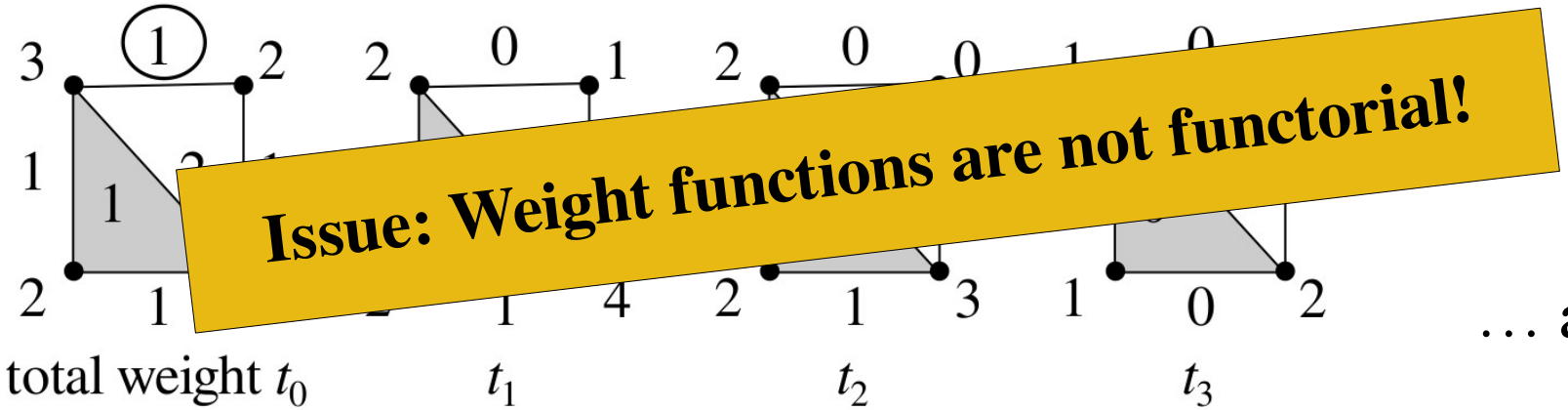


Issue: Weight functions are not functorial!

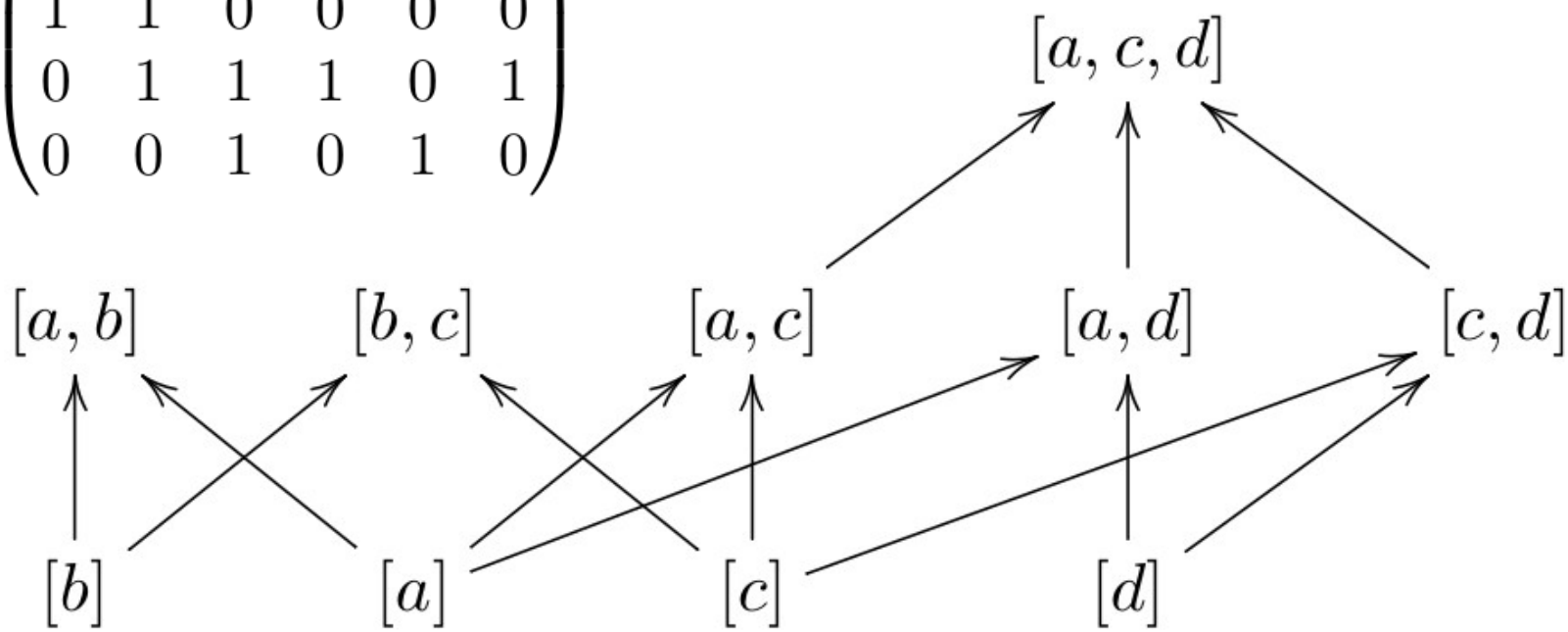
… and so on!

$$r_0 = \begin{pmatrix} \end{pmatrix} \qquad r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 10 \\ 11 \\ 01 \\ 00 \end{pmatrix} \qquad r_3 = \begin{pmatrix} 101 \\ 110 \\ 011 \\ 001 \end{pmatrix}$$

Reorganizing the data using posets

- Draw the Hasse diagram for face poset of the Dowker complex (arrows indicate simplicial inclusions)

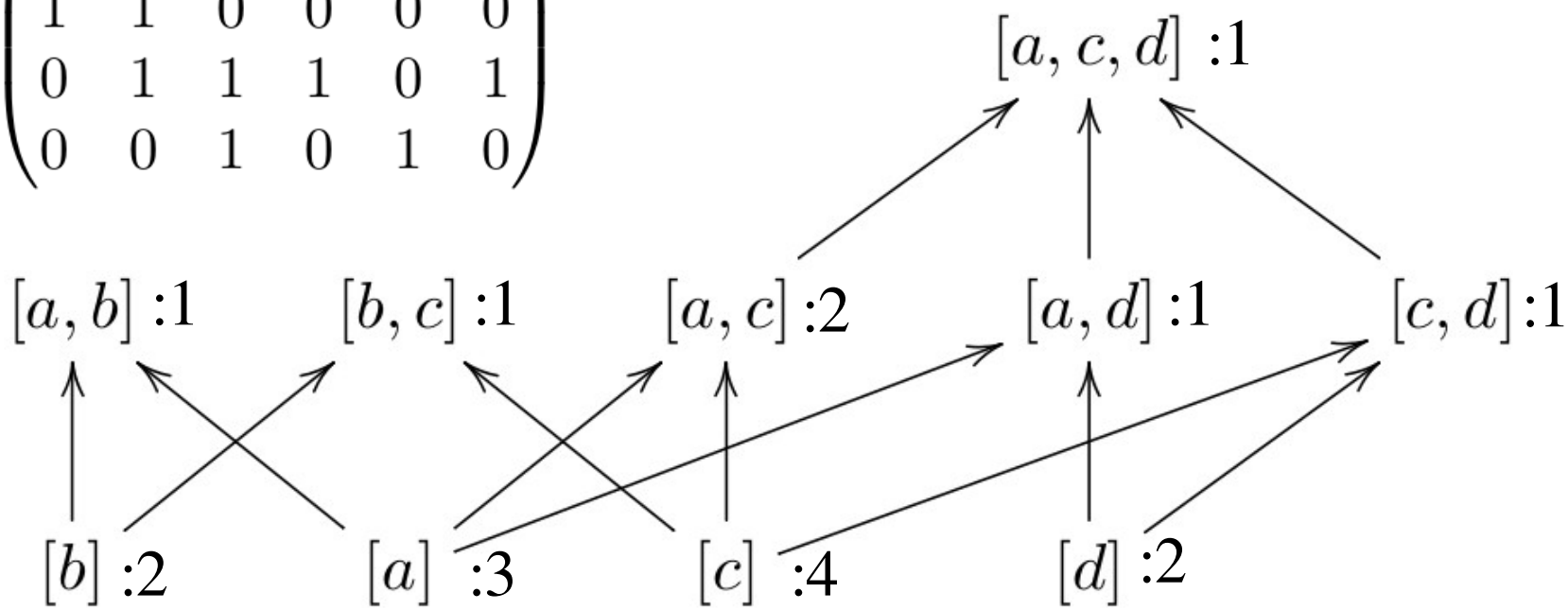
$$
\begin{array}{cc}
 & \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = Y \\
X = \begin{array}{c} \{a, \\ b, \\ c, \\ d\} \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$



Reorganizing the data using posets

- Label the directed graph with total* weights for a lossless representation up to **Rel** isomorphism

$$
\begin{array}{cc}
 & \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = Y \\
X = \begin{array}{c}\{a,\\ b,\\ c,\\ d\}\end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$



*Differential weights seem to be more useful in practice, though…

Dowker cosheaf representation

- But instead of total weights, we can just list the columns!

$$X = \{a, b, c, d\} \quad \begin{array}{c} \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = Y \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

If we want the arrows to be functions, we have to reverse them… They become set inclusions.
The diagram is actually a *cosheaf*

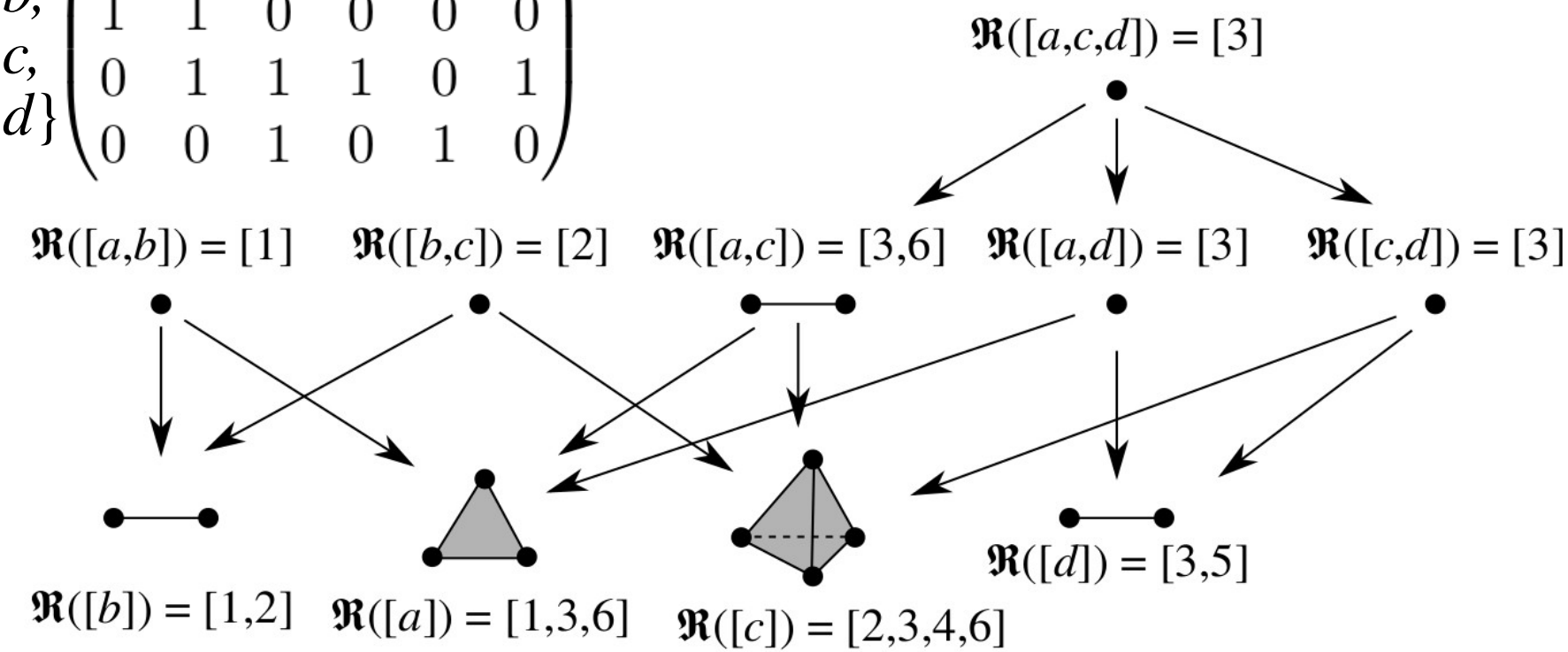
$$\begin{tikzcd}
 & & \mathfrak{R}^0([a,c,d])= \{3\} \arrow[dl] \arrow[d] \arrow[dr] & & \\
\mathfrak{R}^0([a,b])= \{1\} \arrow[d] \arrow[dr] & \mathfrak{R}^0([b,c])= \{2\} \arrow[dl] \arrow[dr] & \mathfrak{R}^0([a,c])= \{3,6\} \arrow[d] \arrow[dl] & \mathfrak{R}^0([a,d])= \{3\} \arrow[d] \arrow[dll] & \mathfrak{R}^0([c,d])= \{3\} \arrow[dl] \arrow[dll] \\
\mathfrak{R}^0([b])= \{1,2\} & \mathfrak{R}^0([a])= \{1,3,6\} & \mathfrak{R}^0([c])= \{2,3,4,6\} & \mathfrak{R}^0([d])= \{3,5\} &
\end{tikzcd}$$

Dowker cosheaf representation

- Each *costalk* is actually a simplex, giving a cosheaf of simplicial complexes on a simplicial complex!

Notation:
CoShvAsc = category of these cosheaves

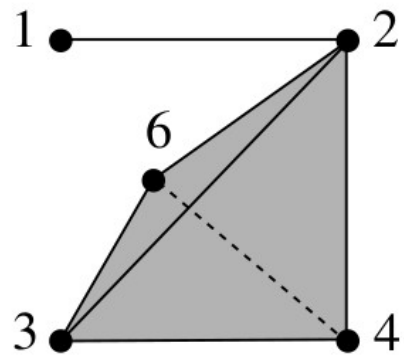
$$X = \{a, b, c, d\} \quad \begin{array}{c} \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = Y \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

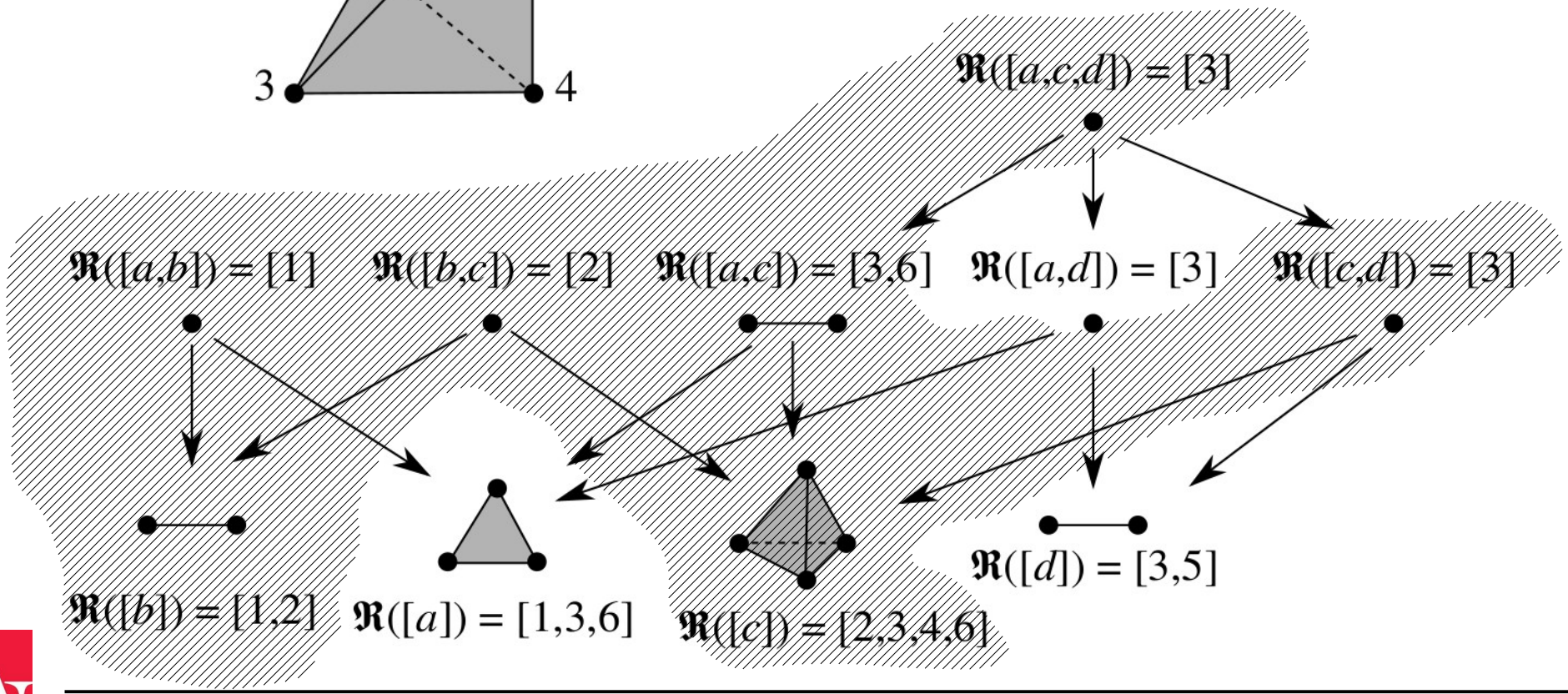


Cosections may be arbitrary complexes

- Cosections on any *open set* are computed by gluing

(upwardly closed in the Hasse diagram = downwardly closed in the cosheaf diagram)





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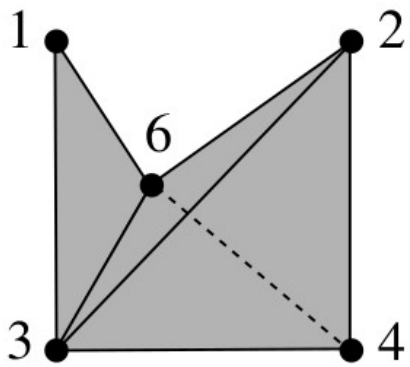
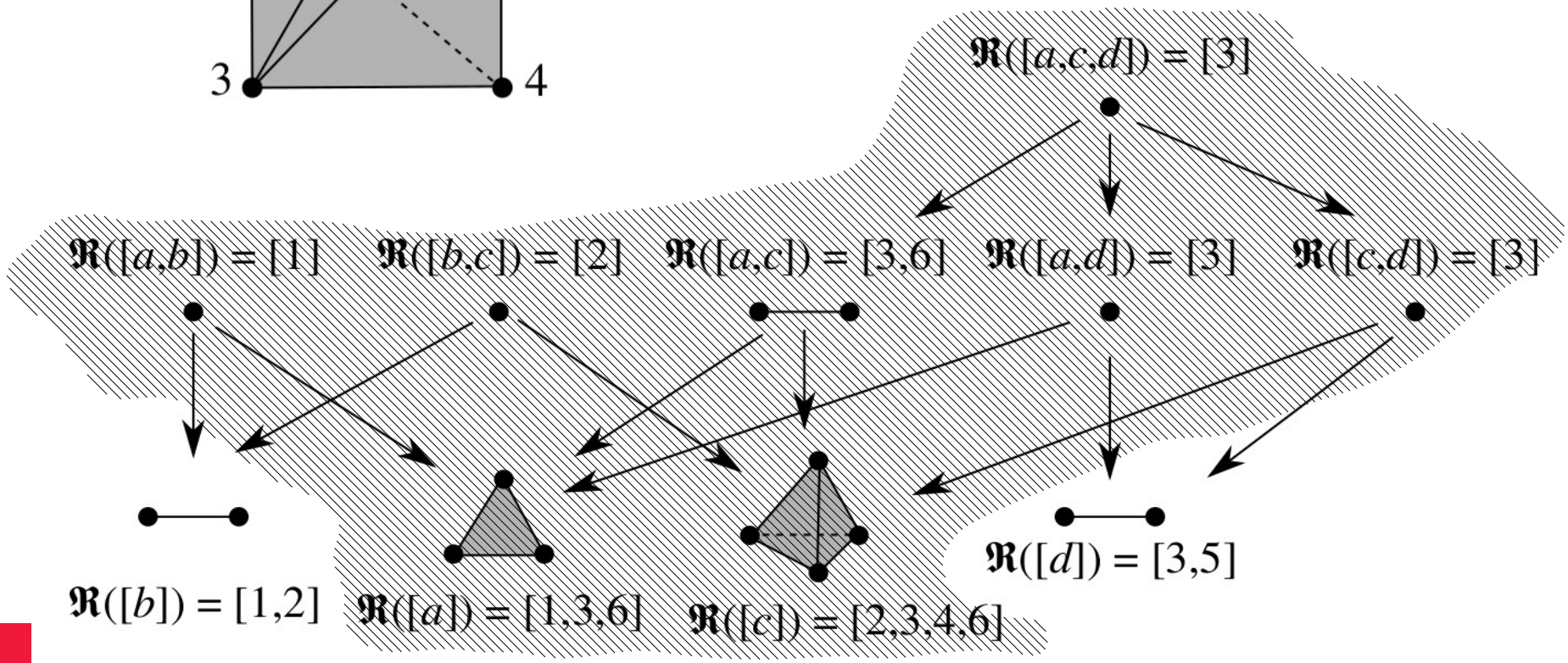
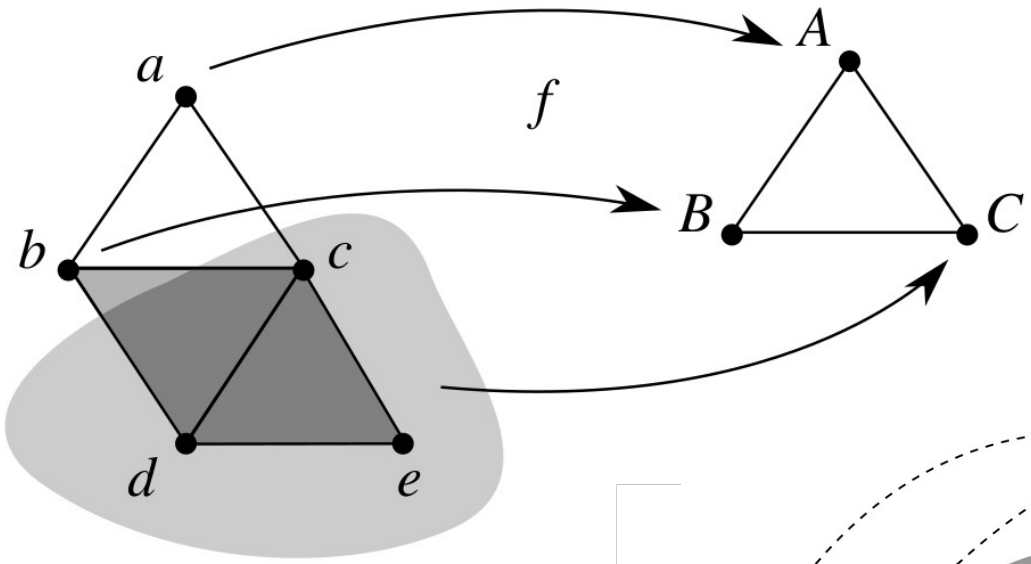


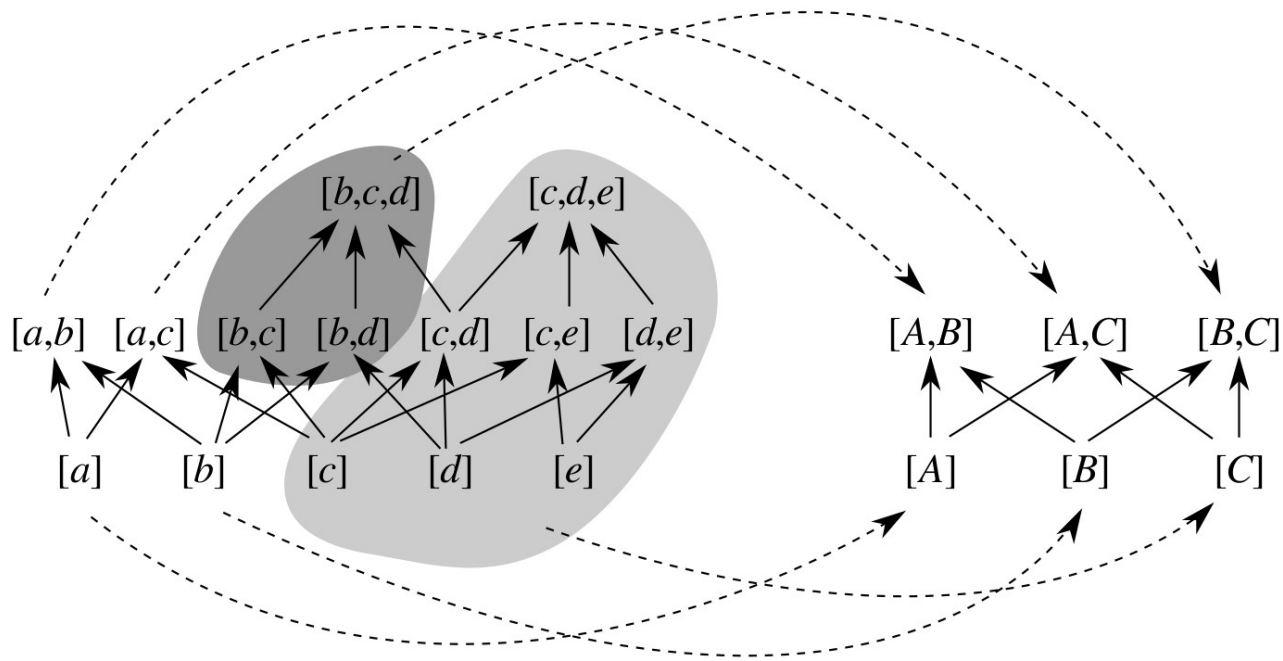


The Dowker complex is functorial...



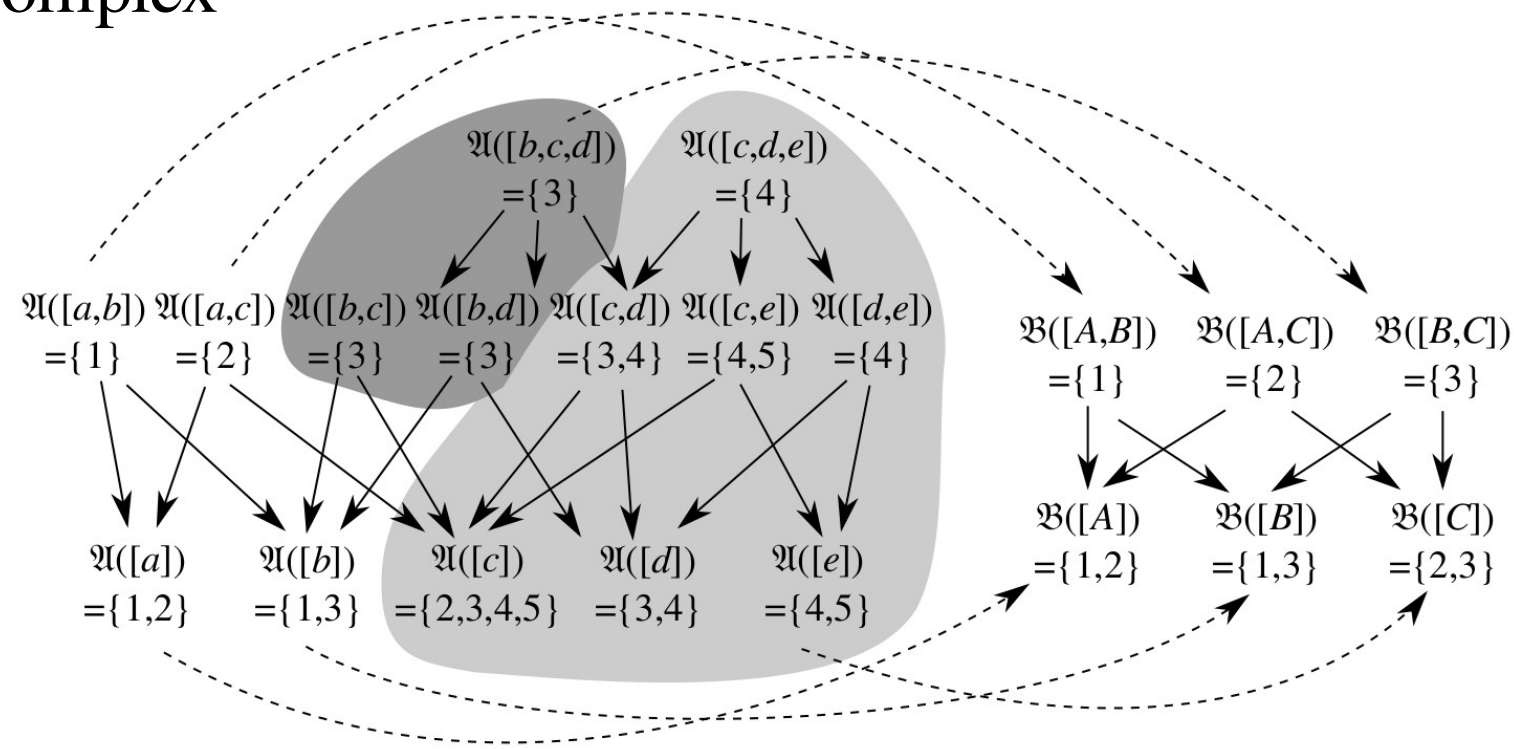
Face partial orders are more useful for most things than simplicial complexes!

For some mysterious reason, this appears to be a folk theorem (?!) It's not hard; exercise for the audience...



… and so is the Dowker cosheaf

- <u>Theorem</u>: The Dowker cosheaf construction defines a faithful covariant functor* **Rel** → **CoShvAsc**.
- The proof mostly follows the functoriality proof for the Dowker complex



*Most authors don't let the base space vary in a (co)sheaf morphism. Beats me why! It's usually better to allow base space maps, and I need them here.

Dowker duality

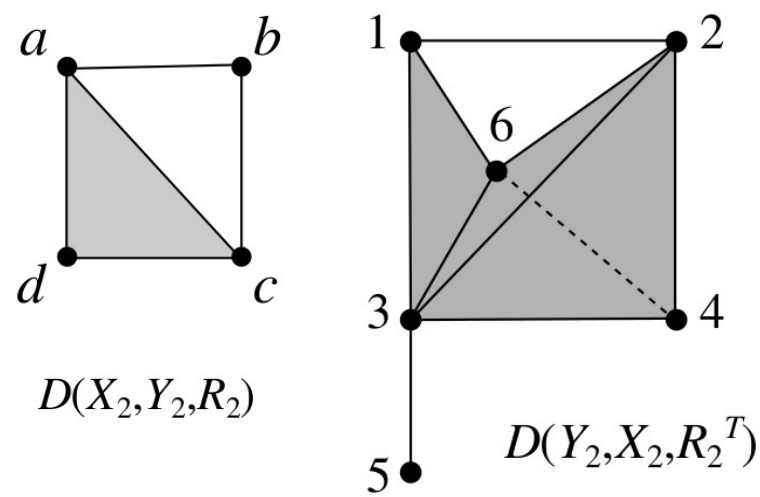
- Dowker's name is attached to these constructions because…
- <u>Theorem:</u> (Dowker*, 1952)

$$H_\bullet(D(X,Y,R)) \cong H_\bullet(D(Y,X,R^T))$$

(The matrix of R^T is the transpose of the matrix of R)

- This has been strengthened to a homotopy equivalence
- Several other generalizations are known

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



$D(X_2,Y_2,R_2)$

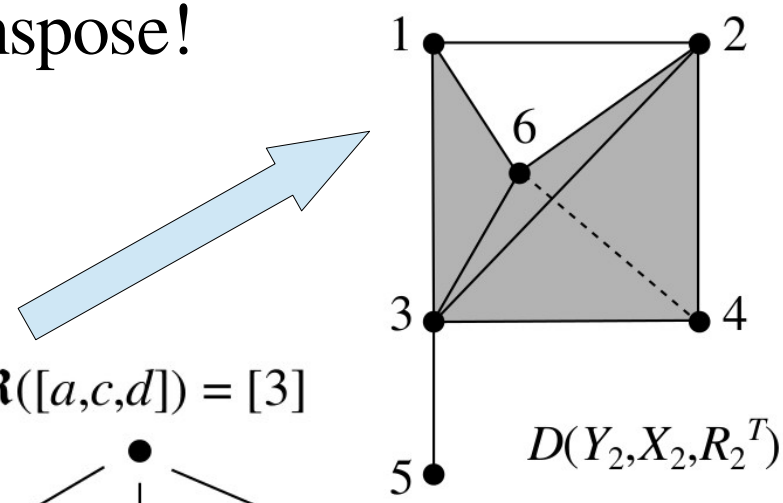
$D(Y_2,X_2,R_2^T)$

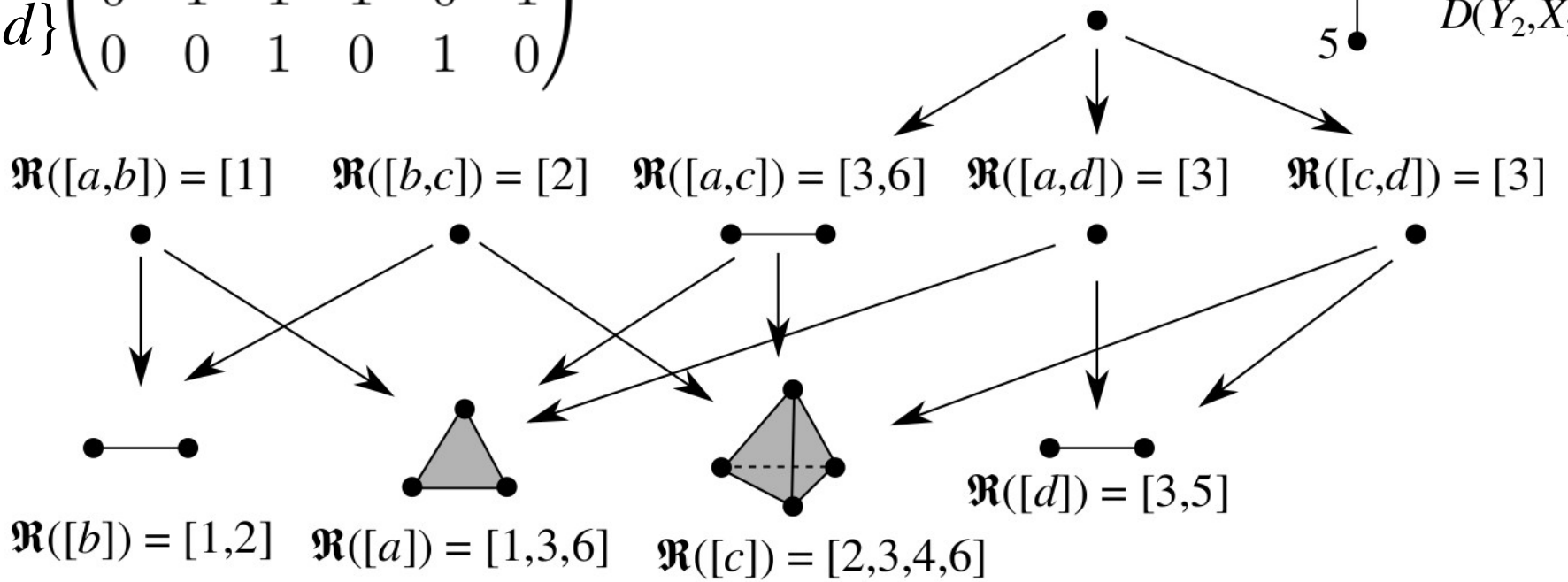
*I found Dowker's paper **very** rough going.
Chowdhury & Mémoli's paper is much easier!

An interesting observation

- The space of global cosections for the Dowker cosheaf is itself a Dowker complex… of the transpose!

$$X = \begin{array}{c} \\ \{a, \\ b, \\ c, \\ d\} \end{array} \begin{array}{c} \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = Y \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$



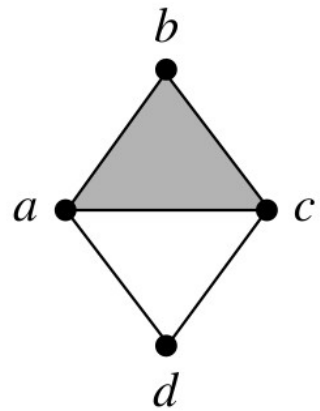


An interesting observation

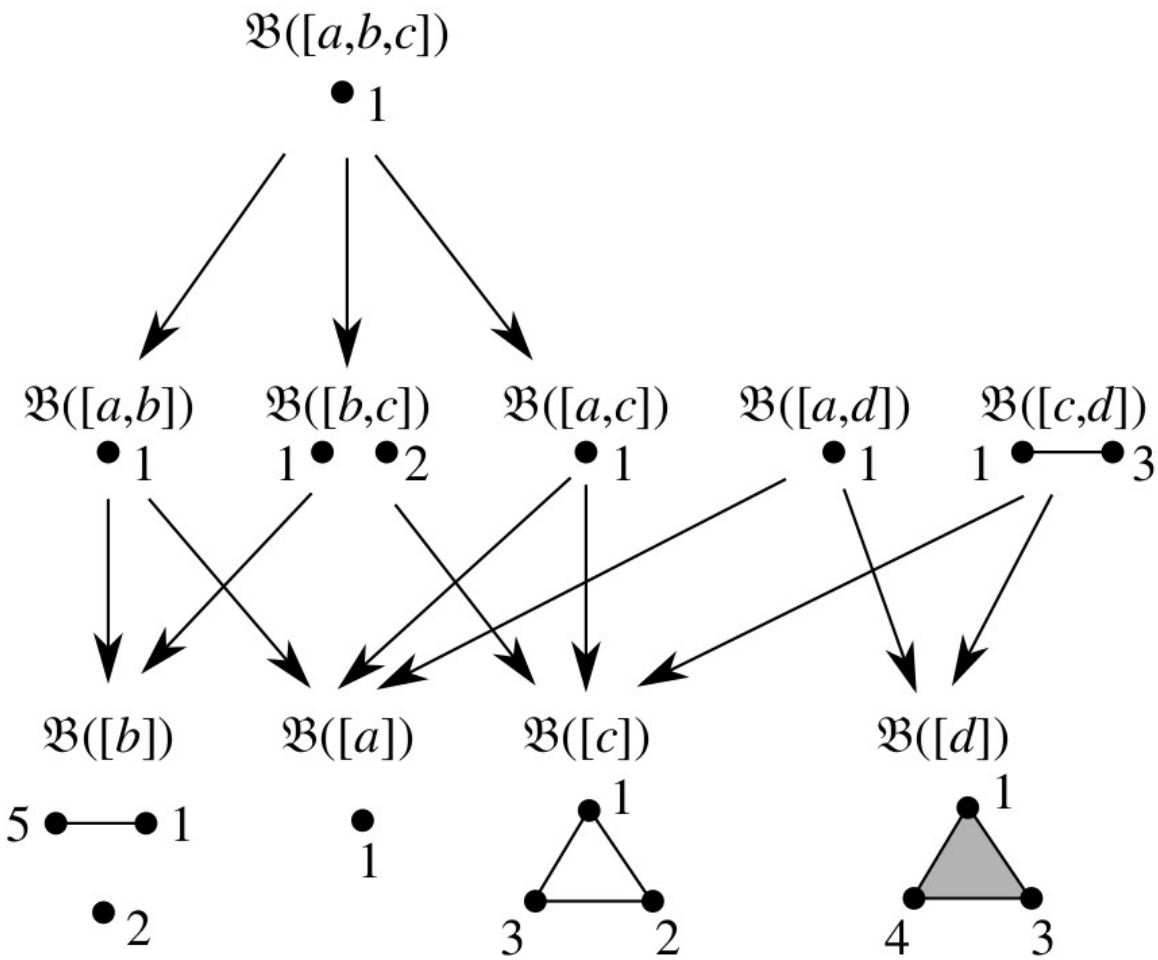
- The space of global cosections for the Dowker cosheaf is itself a Dowker complex… of the transpose!
- The Dowker cosheaf therefore has both the Dowker complex (base space) and its dual (global cosections) baked into it.
- Not only that, this property is **functorial**!
- <u>Theorem</u>: There is a duality functor *Dual* : **CoShvAsc** → **CoShvAsc** that exchanges the base space with the cosections
- The proof, like the others, is an elaborate diagram construction
- Moreover, the definition of *Dual* works for all cosheaves of abstract simplicial complexes

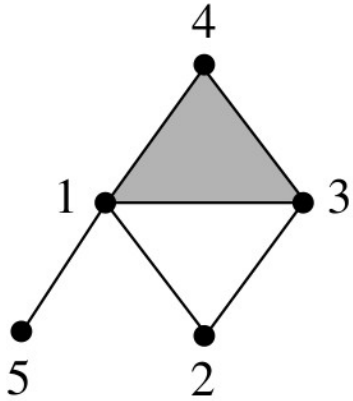
Cosheaf-of-ASC duality functor

- The *Dual* functor acts on non-Dowker cosheaves as well



Base space





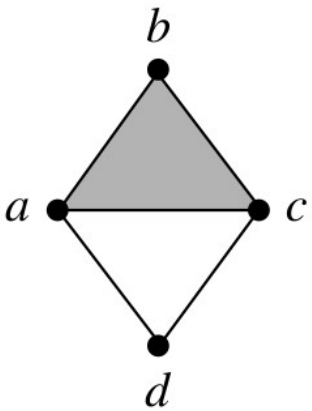
Global cosections

Some random cosheaf of abstract simplicial complexes

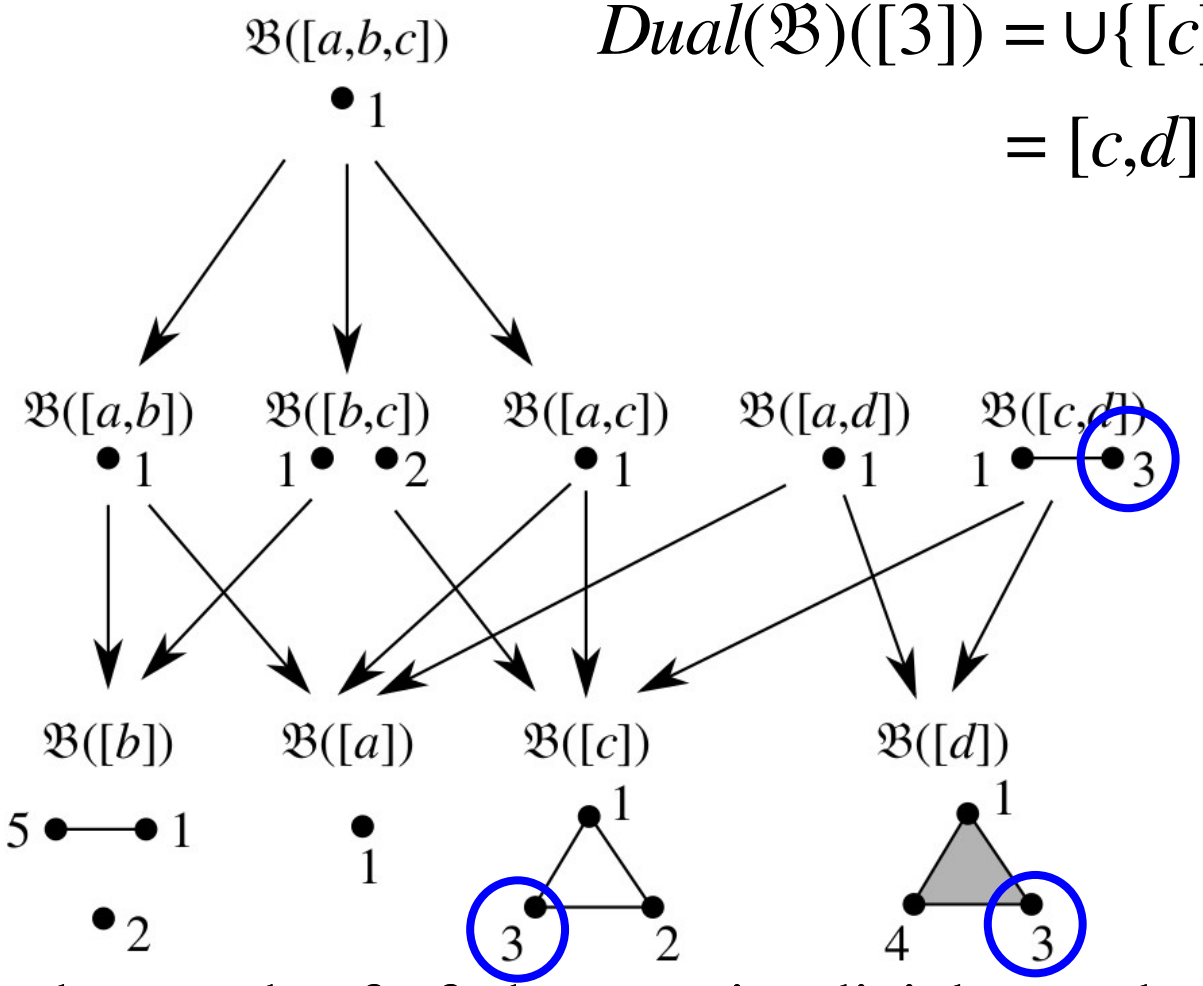
Cosheaf-of-ASC duality functor

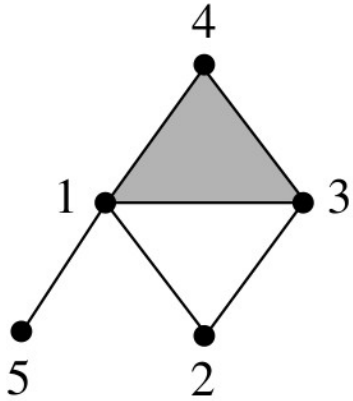
- $Dual(\mathfrak{B})(\sigma)$:= union of all simplices α whose costalks $\mathfrak{B}(\alpha)$ contain σ

$$Dual(\mathfrak{B})([3]) = \cup\{[c],[d],[c,d]\}$$
$$= [c,d]$$



Base space





Global cosections

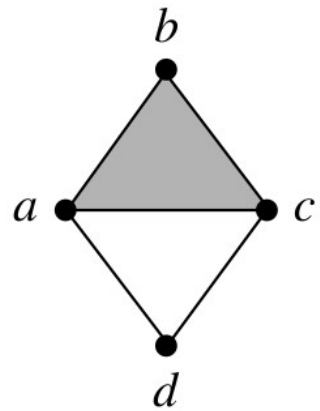
Some random cosheaf of abstract simplicial complexes

Cosheaf-of-ASC duality functor

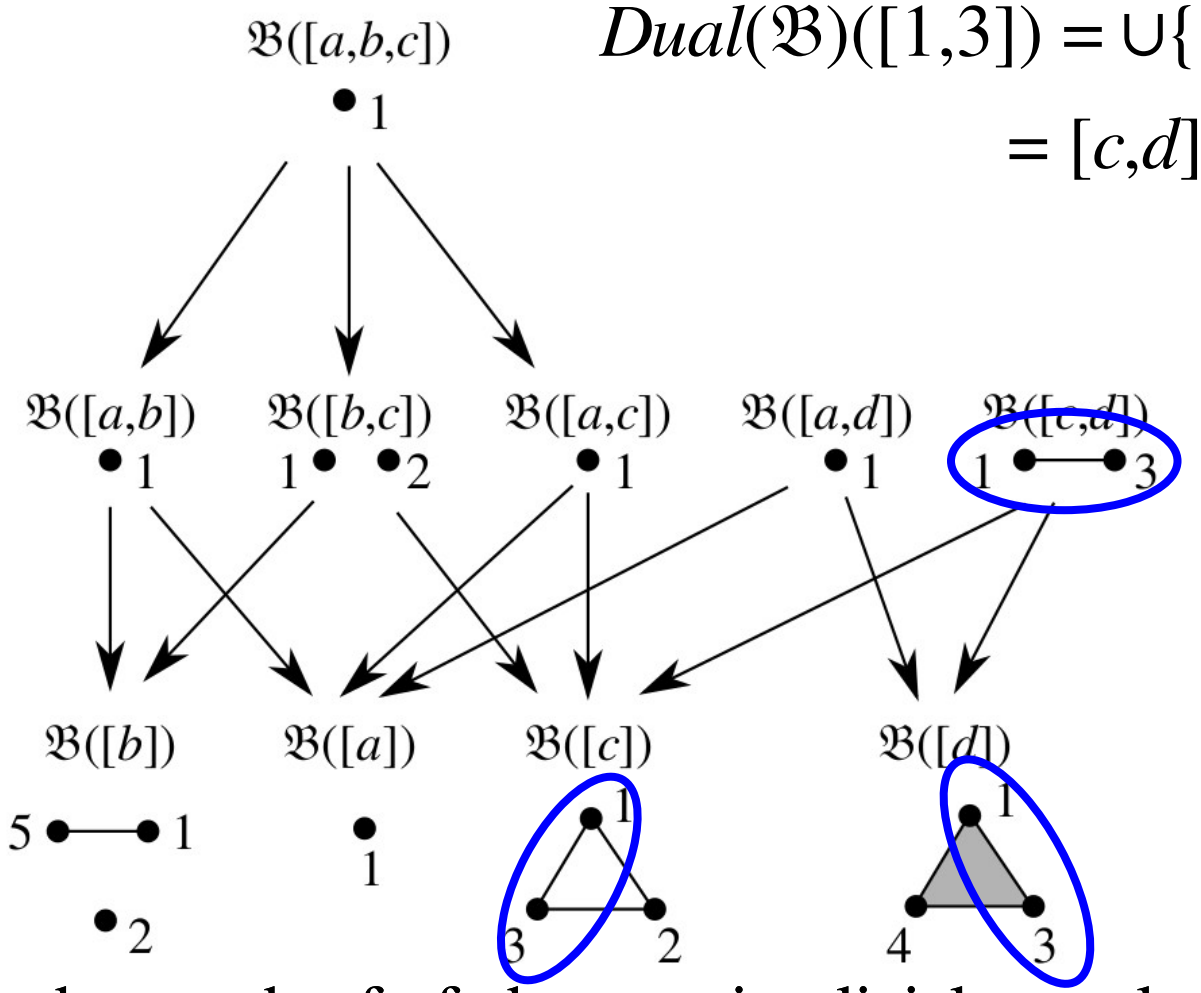
- $Dual(\mathfrak{B})(\sigma)$:= union of all simplices α whose costalks $\mathfrak{B}(\alpha)$ contain σ

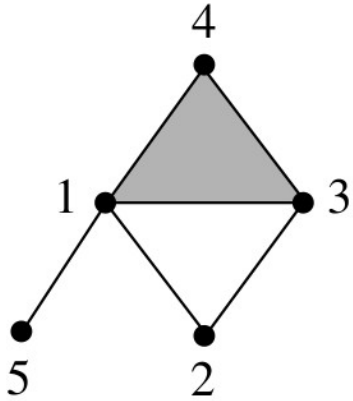
$Dual(\mathfrak{B})([1,3]) = \cup\{[c],[d],[c,d]\}$

$= [c,d]$



Base space



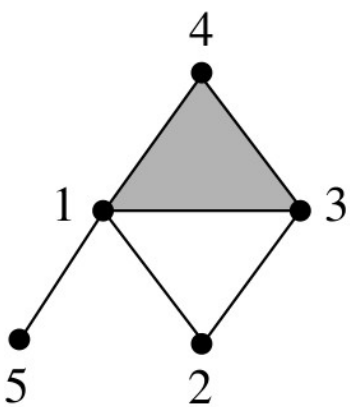


Global cosections

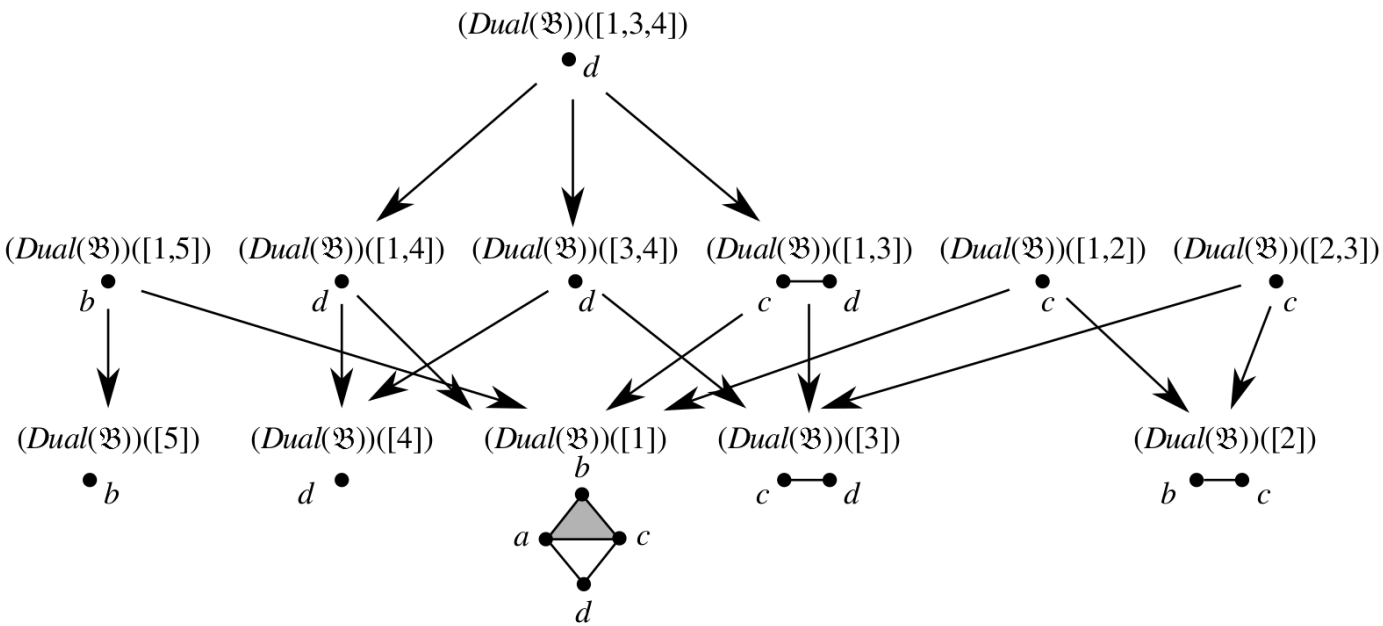
Some random cosheaf of abstract simplicial complexes

Cosheaf-of-ASC duality functor

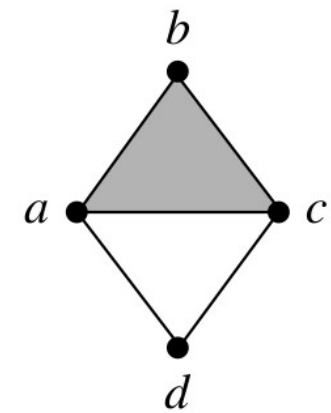
- *Dual* exchanges the base space and space of global cosections
- Note that costalks might not be complete simplices if we aren't working with Dowker cosheaves



Base space



Dual cosheaf



Global cosections

Cosheaf Dowker duality

- <u>Theorem</u>: The space of cosections of the Dowker cosheaf is the Dowker dual of its base space.
  - Briefly, the following functor diagram commutes:

$$\begin{tikzcd}
\mathbf{Rel} \arrow[r, "CoShvRep"] \arrow[d, "Transp"'] & \mathbf{CoShvAsc} \arrow[d, "Dual"] \\
\mathbf{Rel} \arrow[r, "CoShvRep"'] & \mathbf{CoShvAsc}
\end{tikzcd}$$

  - The only thing that needs to be shown is that dualizing a Dowker cosheaf yields a new cosheaf whose costalks are all complete simplices … and a bit of calculation besides.

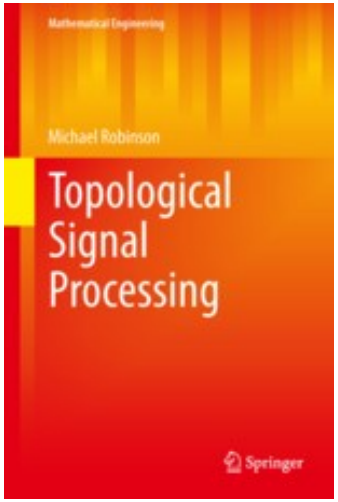
Implications for consensus file formats

- Because of functoriality, any of these Dowker constructions are reasonable tools for studying parser behavior on files
- Functoriality guides the process of summarization
  - Which files are good exemplars of (non)compliance?
  - Which parsers are redundant, or conversely, which have divergent capabilities?
  - Statistics on the values of the weights “makes sense”
- Functoriality guides the process of curating corpora
  - If we test the same set of parsers on two different corpora, how do we compare the results?
- You can consider parser-file relations or file-parser relations without losing anything

Next steps

- We can generalize **Rel** to take poset values rather than booleans
  - This breaks most of the existing technology for filtrations, but **not** our cosheaf constructions
- <u>Theorem</u>: The Dowker cosheaf generalized for poset-valued **Rel** is functorial
  - What can be said about persistence constructions in this case?
- We recently discovered a pseudometric on **Rel**, which gives it a topology
- <u>Theorem</u>: The total and differential weight functions are both continuous with respect to this topology on **Rel** under the sup-norm
  - Chowdhury & Mémoli's starting point is a filtration on the Dowker complex, which the total weight provides
  - This suggests that *interleavings* of cosheaves might work. (Cool! I have some results on sheaf-based general interleavings)
- Explore more of the category theoretic structure of **Rel**
  - I suspect it has at least two distinct (co)products and more category theoretic delights!

To learn more...



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Relevant preprints:

https://arxiv.org/abs/2003.00976

https://arxiv.org/abs/2005.12348

Software:

https://github.com/kb1dds