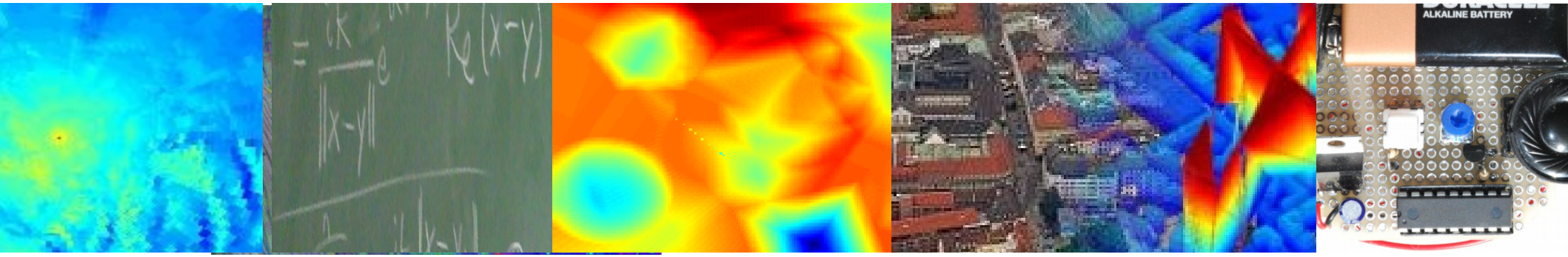
What is topology?



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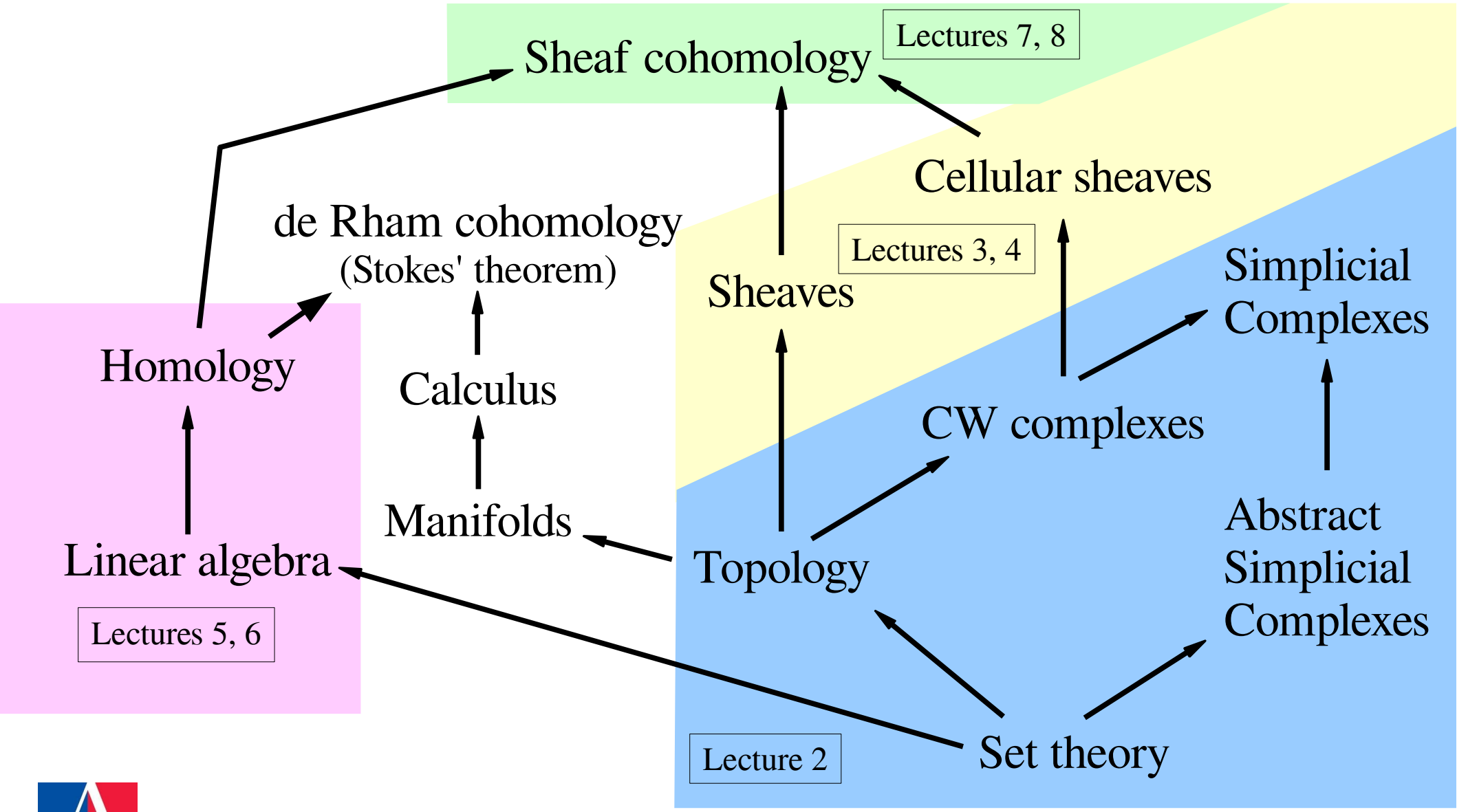
SIMPLEX Program

Mathematical dependency tree



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Session objectives

- What is a topological space?
  - What kinds of spaces are there?
- How to represent topological spaces effectively for data fusion problems
  - Simplicial complex models, and how they interface with topology
- Encoding data as a topological space

What is topology?

=

Topology is the study of spaces under continuous deformations

What is topology?

Not this!
This is
TopoGRAPHY!

(Thanks USGS!)

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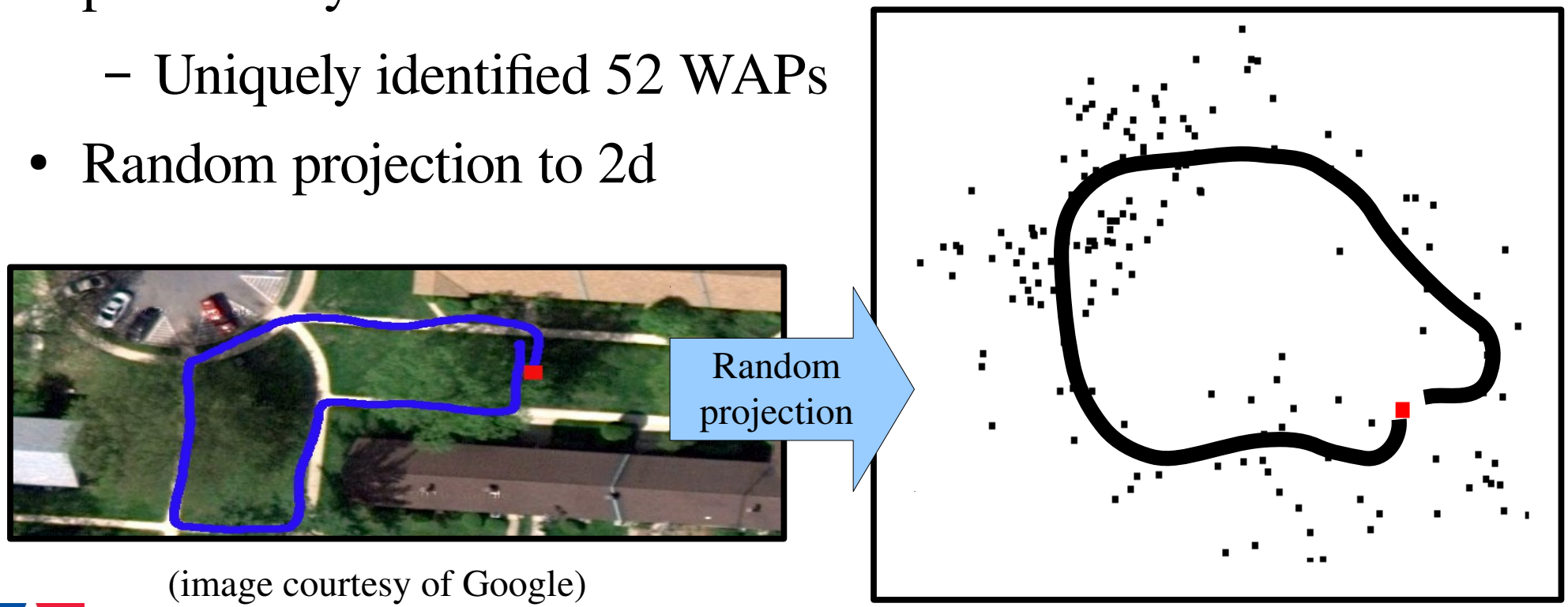
Topological imaging

- Cellphone used to record signal level of 802.11 access points near several apartment buildings
- Signal level (dBm) and station MAC address recorded periodically
  - Uniquely identified 52 WAPs
- Random projection to 2d

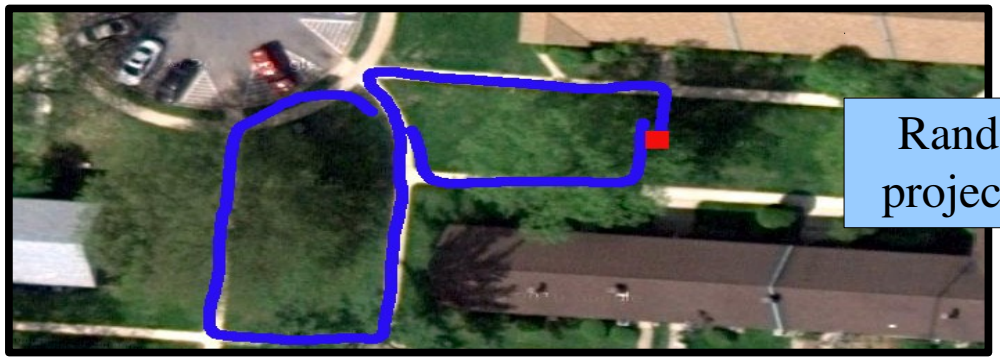


(image courtesy of Google)

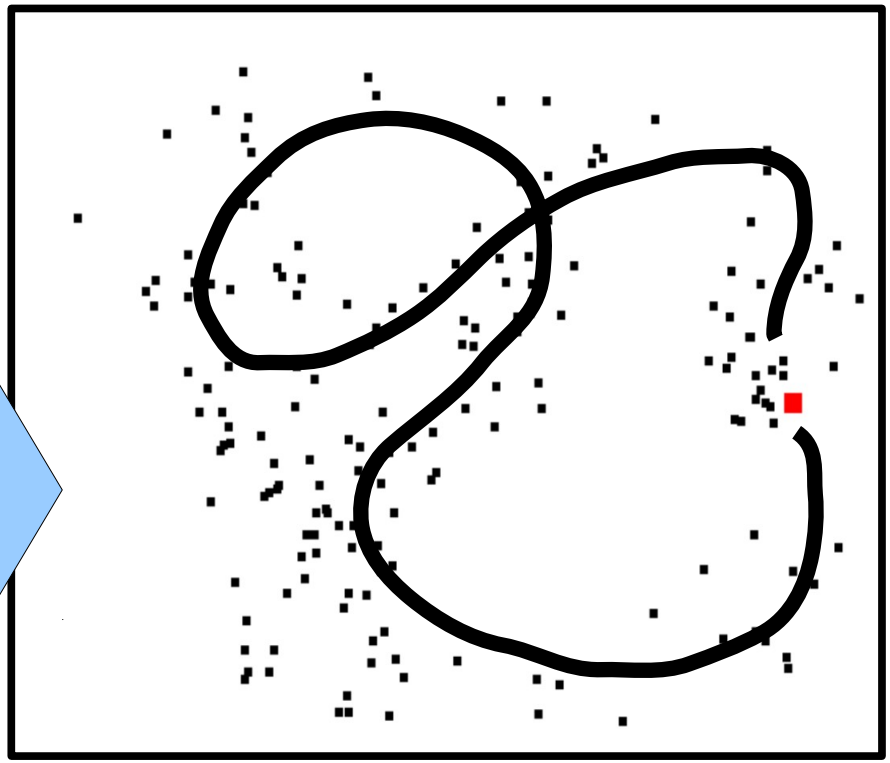
Software credit: Daniel Muellner and Mikael Vejdemo-Johansson

Topological imaging

- Goal: measure environment and targets with minimal sensing and opportunistic sources
- Key theoretical guarantees proven
- First generation algorithms
  - Simulated extensively
  - Validated experimentally



Random projection



(image courtesy of Google)

Software credit: Daniel Muellner and Mikael Vejdemo-Johansson

What kind of space?

<u>General topological space</u>

- More general (obviously!)
- Very traditional
- But admits pathological cases

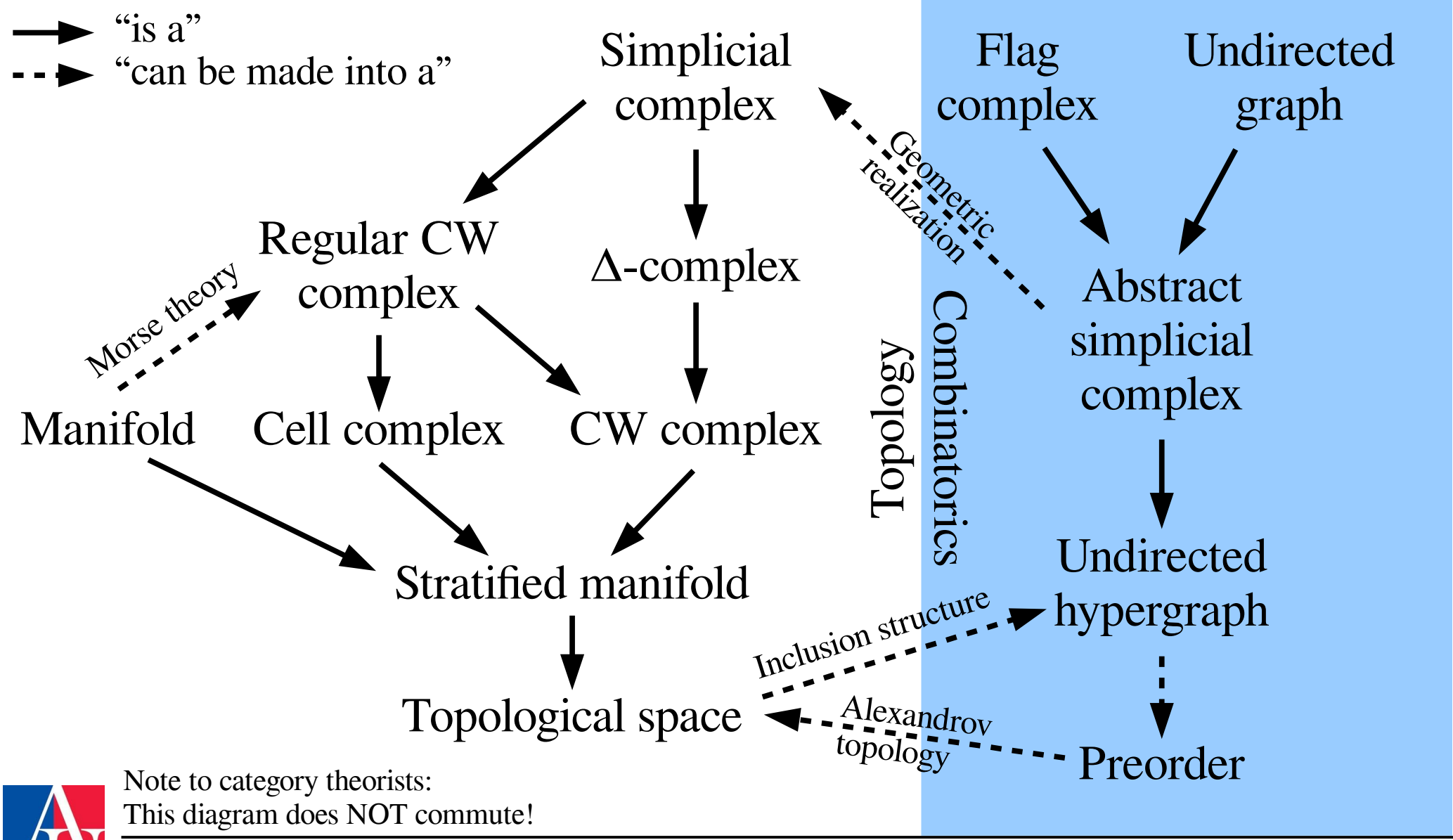
<u>Manifolds</u>

- Very restricted, but this aids estimation
- Calculus!

<u>Cellular space</u>

- Less general (and less pathological)
- Good for computation
- Less general (and less expressive)
- More constraints

Some (mostly) topological spaces



Note to category theorists:
This diagram does NOT commute!

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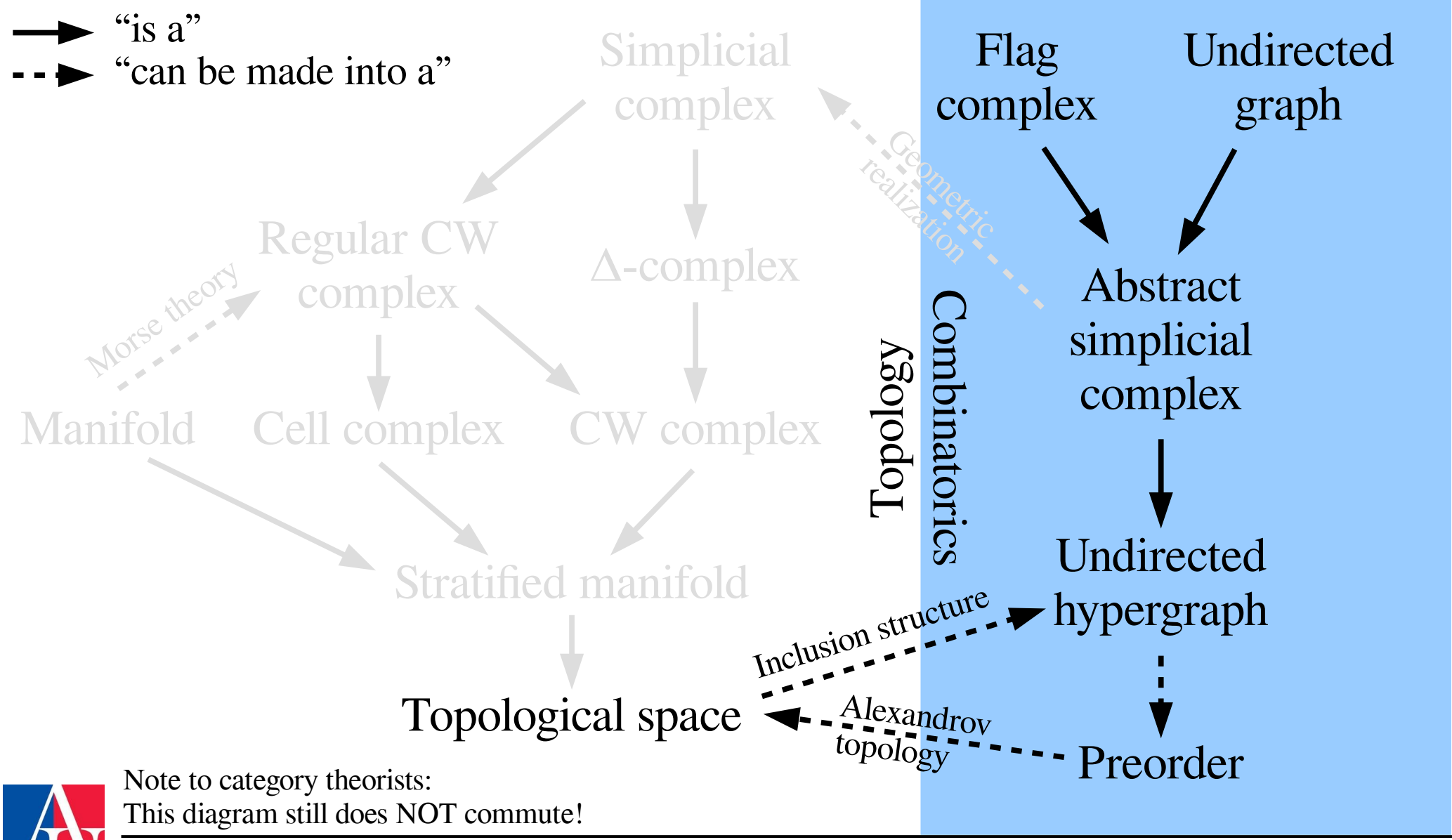
Spaces of interest for this tutorial



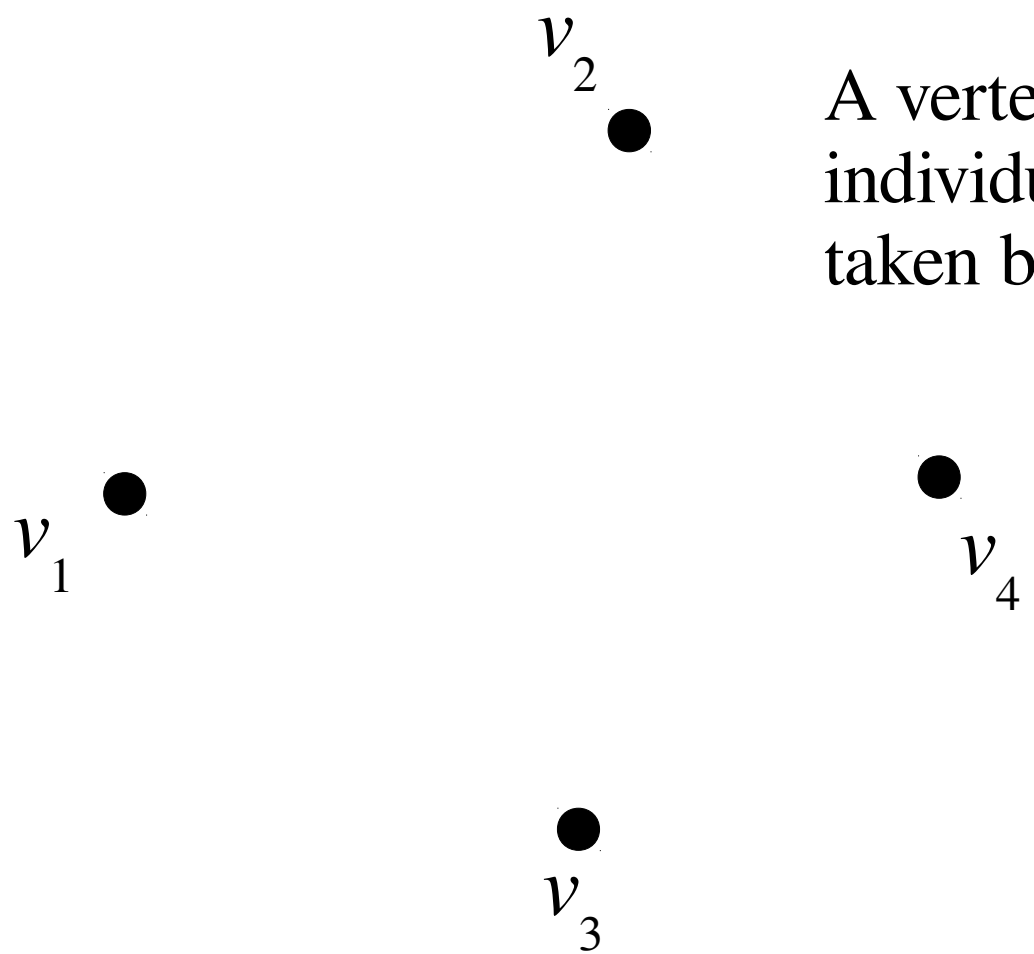
Note to category theorists:
This diagram still does NOT commute!

High level intuition first...

… then the precise details

Simplicial complexes

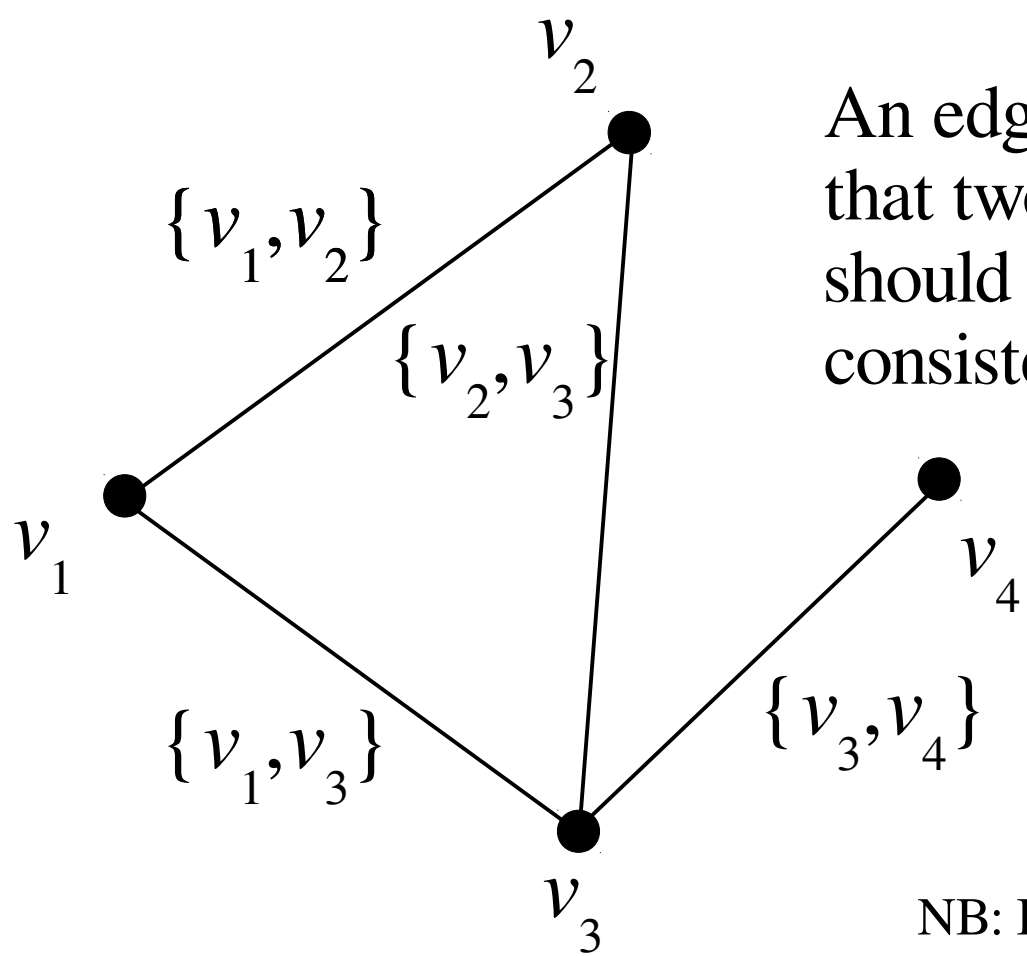
- A simplicial complex is a collection of *vertices* and ...



A vertex represents an individual measurement taken by a sensor

Simplicial complexes

- … *edges* (pairs of vertices) and ...



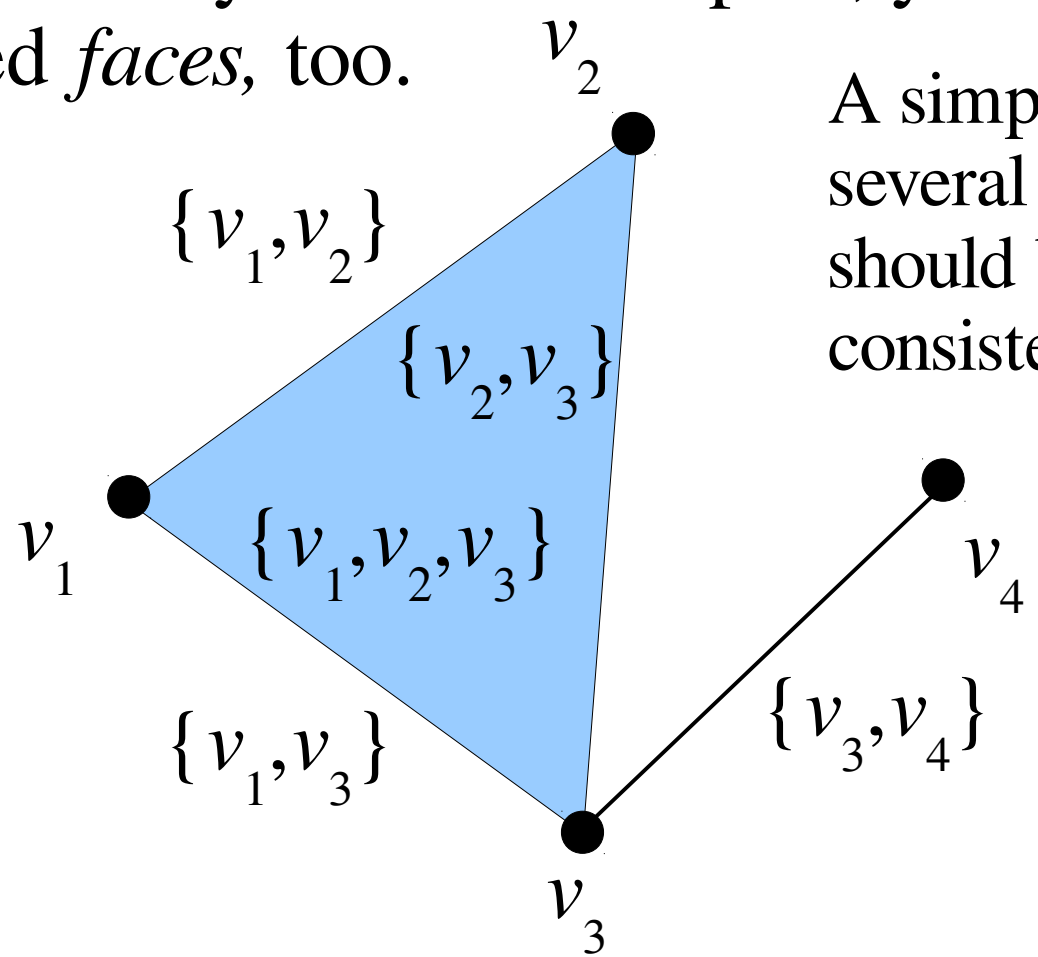
An edge represents the fact that two measurements should be tested for consistency

Usually people call this a *graph*; I will too

NB: I got lazy… v_1 should be $\{v_1\}$

Simplicial complexes

- ... higher dimensional *simplices* (tuples of vertices)
- Whenever you have a simplex, you have all subsets, called *faces*, too.



A simplex represents that several measurements should be tested for consistency

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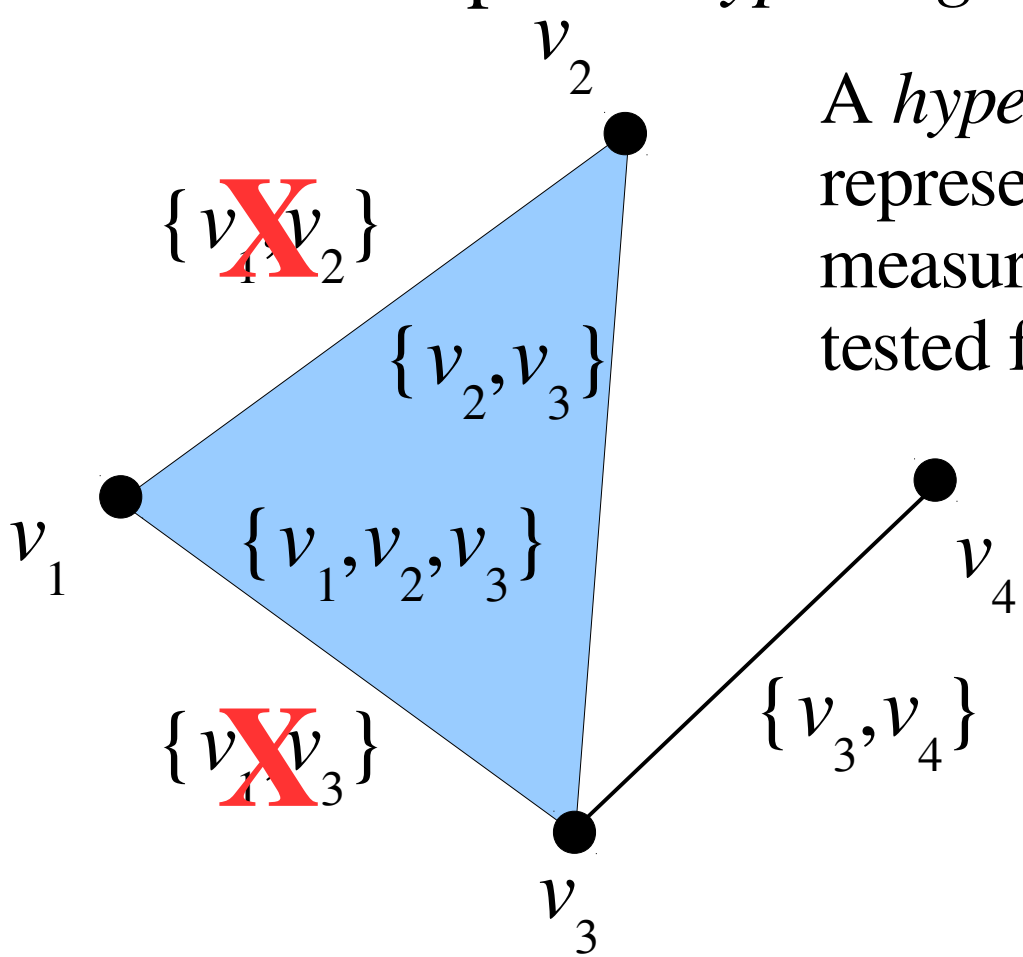
Undirected hypergraphs

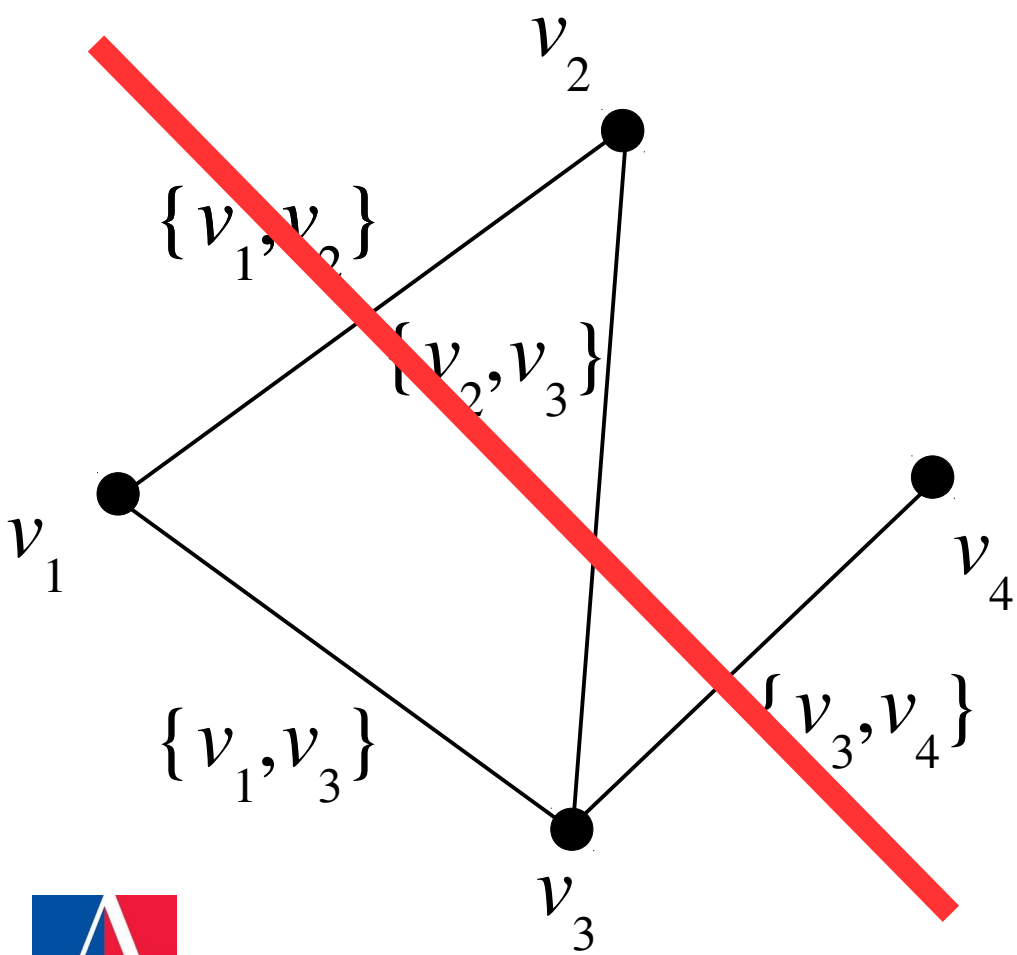
- ... might have simplices that are missing some *faces*
- … and we call simplices *hyperedges* as a reminder

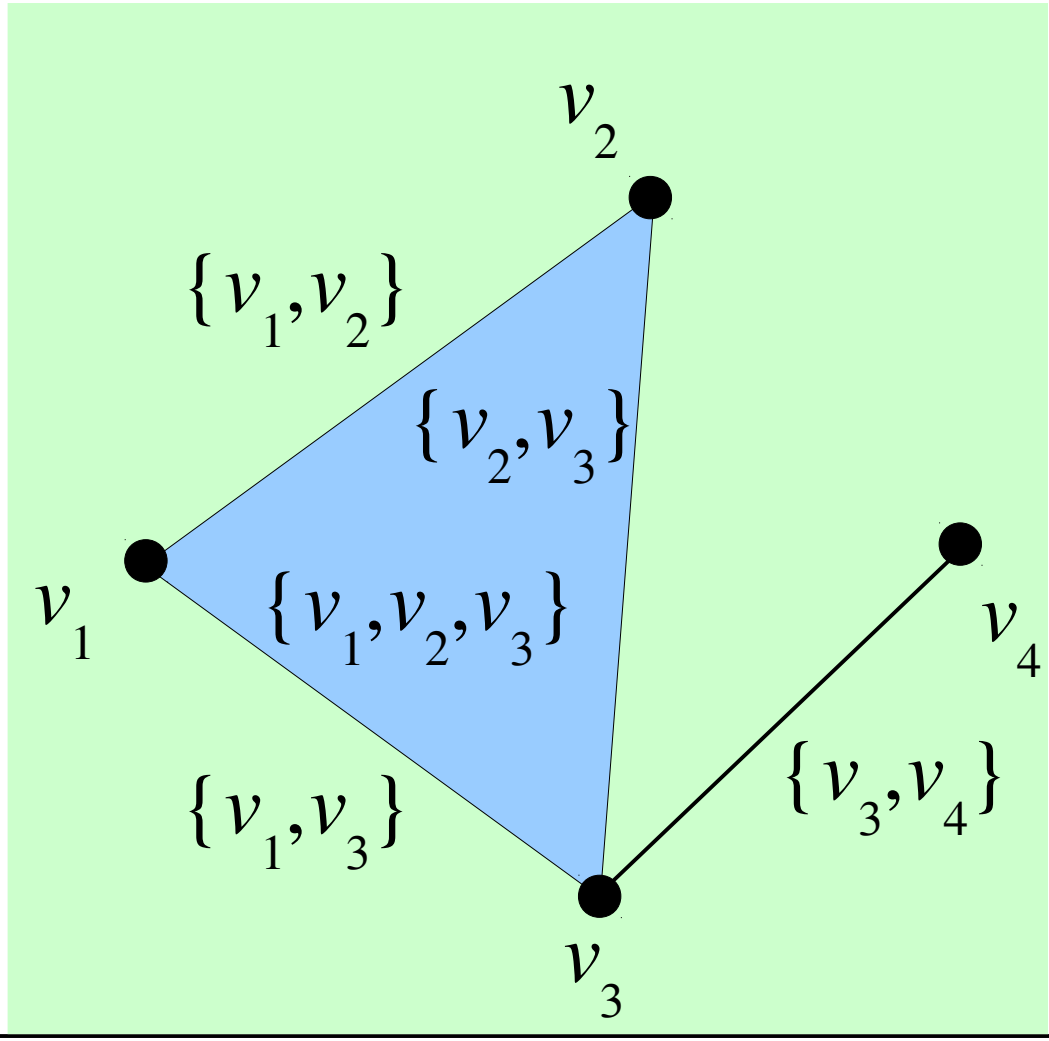


A *hyperedge* still represents that several measurements should be tested for consistency

Flag complexes

- All *cliques* – an edge between every pair of vertices – are simplices, too





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Helpful taxonomy

All cliques are simplices

Flag complex

Undirected graph

Vertices and edges only

Simplices are defined by their vertices, simplices always have all their faces

Abstract simplicial complex

Easier to compute invariants

Harder to compute invariants

Hyperedges defined by vertices, but other than that, anything goes...

Undirected hypergraph

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Now the details!

Abstract Simplicial Complex

Given a set V, which we'll call *vertices*, an abstract simplicial complex is a collection X of subsets of V satisfying the property:

If $a \in X$ and $b \subseteq a$ then $b \in X$.

(If you ignore this property, then X is a *hypergraph*.)

Elements of X are called *simplices*

- A simplex $a = \{a_0, a_1, \ldots a_n\}$ is said to have *dimension n*
- A maximal simplex is called a *facet* or a *toplex*
- Dimension 0 = *vertex*, dimension 1 = *edge*

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Flag complexes and graphs

- If X only has vertices and edges, it's called a *graph*
- A *clique* in a graph is a subgraph in which every pair of vertices has an edge between them
- If all cliques are also simplices, then X is a *flag complex*

- Flag complexes are often better choices than graphs because they capture higher dimensional (multi-way) relationships

Topology

- A way to describe neighborhoods of simplices
- This is really good for describing where a piece of data is valid – near the sensor, usually
- There's a natural way to do this completely generally, and it's very handy mathematically
  - It leads to computational trickiness, though
  - We'll use the *Alexandrov topology* when needed (usually for hypergraphs) and avoid the general definition
- Fortunately, there's usually a way to go between topology and simplicial complexes

Topology, precisely

If X is a set, then a *topology* T on X is a collection of subsets of X satisfying four axioms:

1. The empty set is in T
2. X is in T
3. If U is in T and V is in T then $U \cap V$ is in T
4. All unions of elements of T are in T

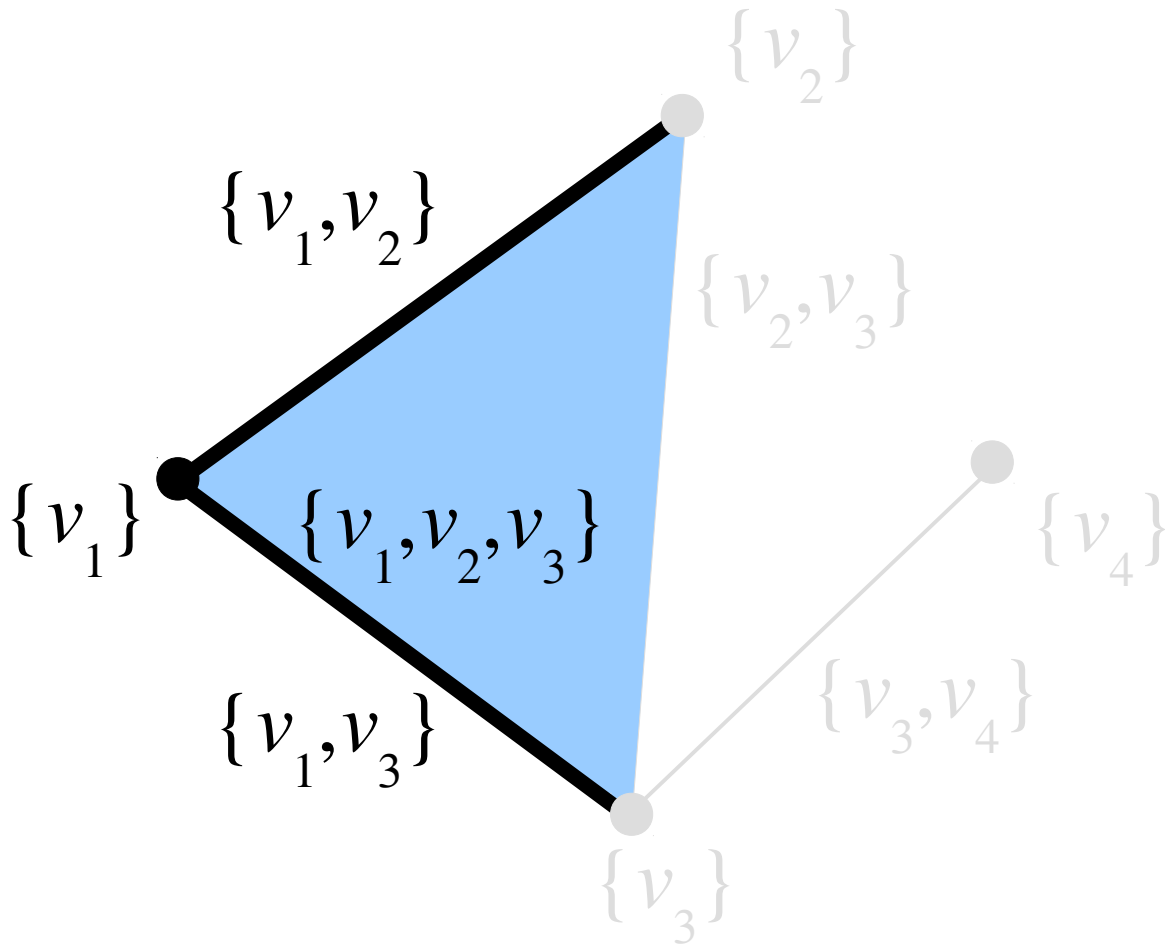
The elements of T are called the *open sets* for the topology

The pair (X,T) is called a *topological space*

Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it

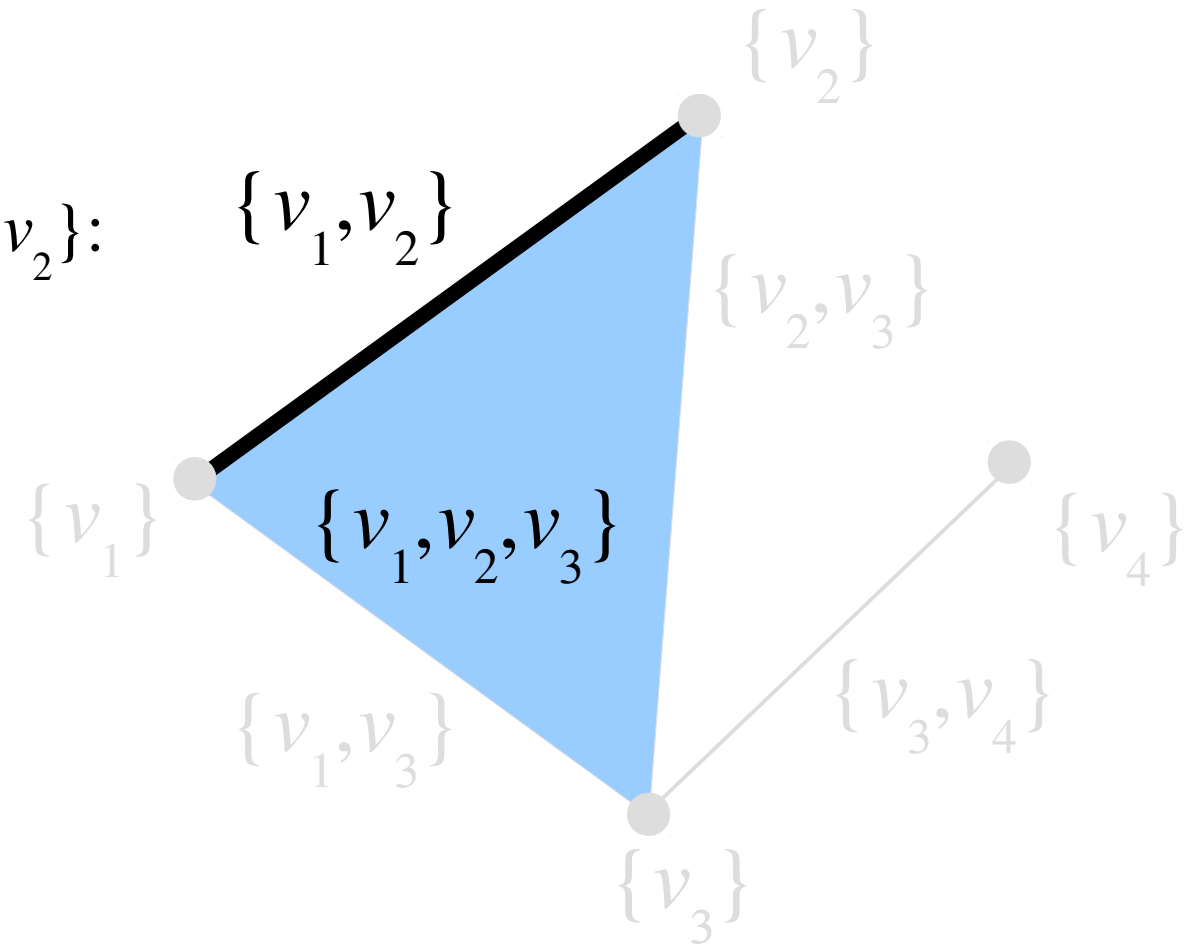
Star over v_1:



Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it

Star over $\{v_1, v_2\}$:



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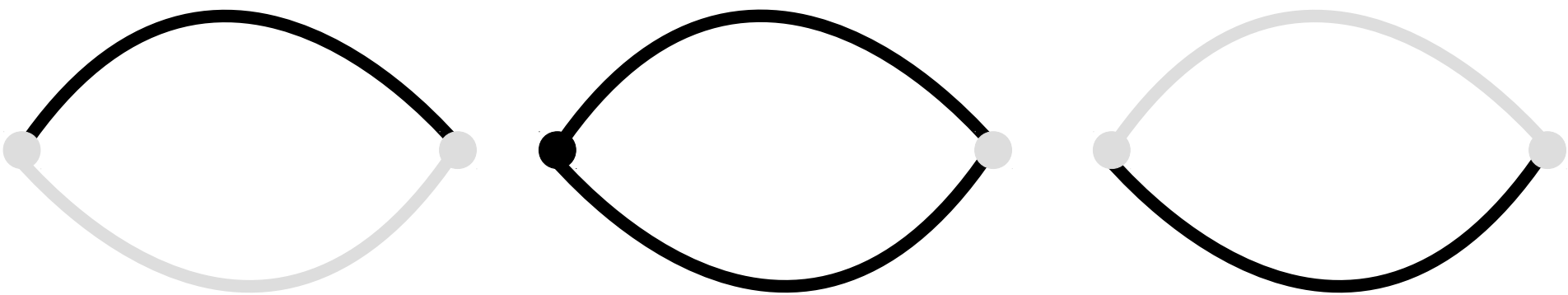
Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it
- <u>Proposition</u>: The collection of all possible stars and all possible unions of stars forms a topology for the set of simplices
- It's called the *Alexandrov topology*, and it's usually the correct one for data integration problems
- This works for hypergraphs, not just abstract simplicial complexes

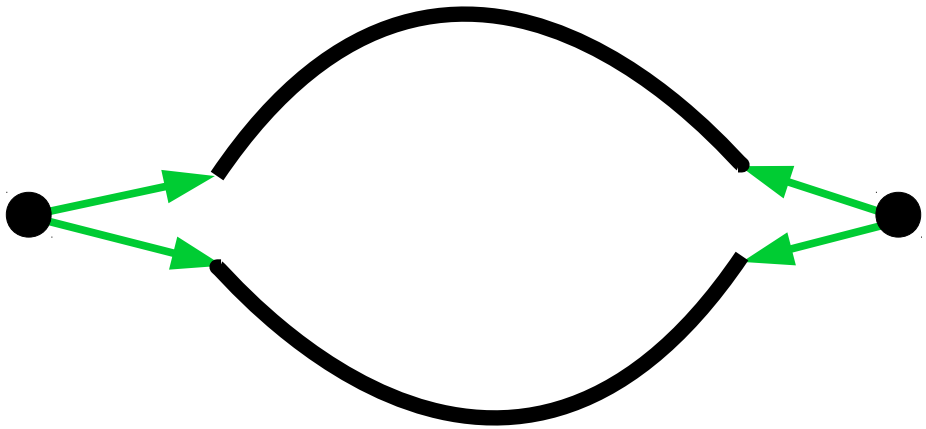
Beyond hypergraphs

- Sometimes you need to duplicate simplices…
  - This isn't a problem for the topology, but it violates our combinatorial construction thus far
  - This usually leads to **multi**graphs and **multi**hypergraphs
- The Alexandrov topology still works, though, and this causes no computational problems in the end



Beyond hypergraphs

- What's important is the set of *attachments* between simplices
  - These form a partial order for the simplices, via set inclusion
  - Vertices are subsets of edges, etc.

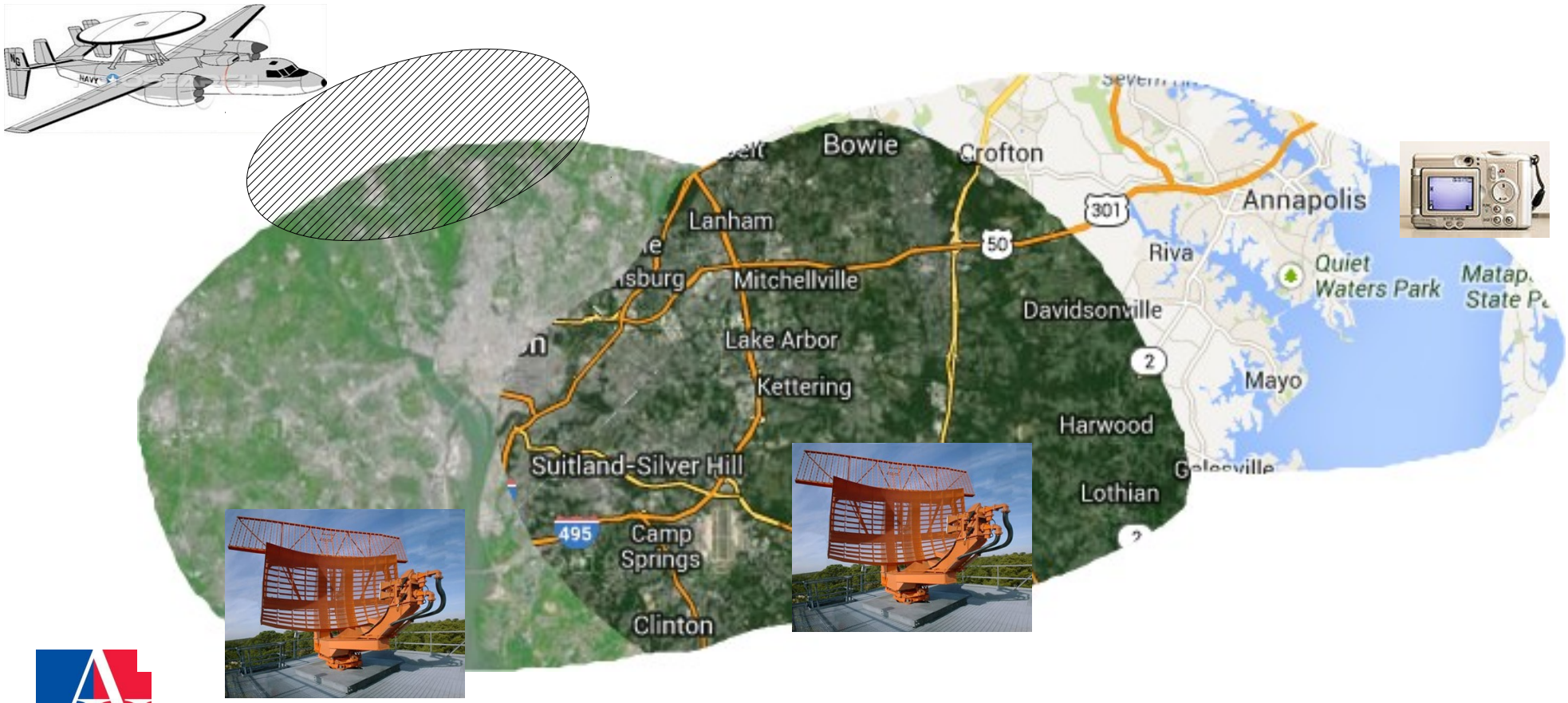


Starting with a topology

- There's usually a physical region associated to a sensor where it's gathering information
  - It's usually good practice to assert that this is an open set in some appropriate topology, often the usual topology for Euclidean space

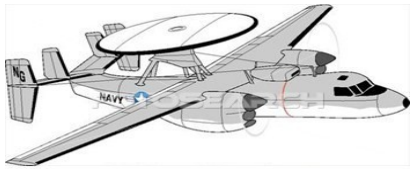
Recall: Multi-INT tracking

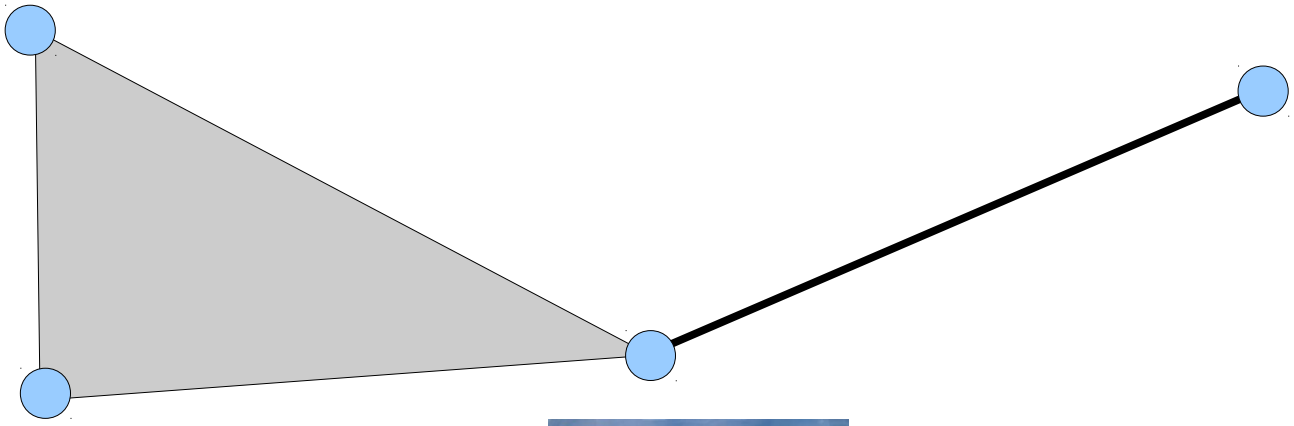
- Each sensor has a region associated to it
  - Should be an open set in Euclidean space for these sensors; a good guess for physical sensors



Topologizing the sensor space

- We now know this is a simplicial complex!
- But how did we get it?





Nerve of an open cover

Let (X,T) be a topological space and $\mathscr{U}=\{U_1,U_2,\ldots\}$ be a collection of open sets whose union is X (an *open cover*)

The *nerve* $N(\mathscr{U})$ is the abstract simplicial complex with

- The set of vertices given by $\mathscr{U}$, and
- Whenever $U_a \cap U_b \cap \ldots$ is nonempty, $\{U_a, U_b, \ldots\}$ is a simplex in $N(\mathscr{U})$

A crucial theorem (Leray theorem) links $N(\mathscr{U})$ to the topological structure of (X,T)

Next up...

- Lunch!
- Next lecture: Define sheaves!

Further reading...

- Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002
  - http://www.math.cornell.edu/~hatcher/AT/AT.pdf
- James Munkres, *Topology*, Pearson, 2000
- Gurjeet Singh, Facundo Mémoli and Gunnar Carlsson, "Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition," *Eurographics Symposium on Point Based Graphics, European Association for Computer Graphics*, 2007, pp. 91–100
- Gunnar Carlsson, "Topology and Data," *Bull. Amer. Math. Soc*. 46 (2009), 255–308
- Dai Tamaki, "Cellular stratified spaces"
  - http://arxiv.org/abs/1106.3772
  - http://arxiv.org/abs/1111.4774
- Michael Robinson and Robert Ghrist "Topological localization via signals of opportunity," *IEEE Trans. Sig. Proc*, Vol. 60, No. 5, May 2012.