Topological localization via signals of opportunity

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***Abstract*—We consider the problems of localization, disambiguation, and mapping in a domain filled with signals-of-opportunity generated by transmitters. One or more (static or mobile) receivers utilize these signals and from them characterize the domain, localize, disambiguate, etc. The tools we develop are topological in nature, and rely on interpreting the problem as one of embedding the domain into a sufficiently high-dimensional space of signals via a signal profile function. A variety of different scenarios are addressed, including varying signal types (TOA, TDOA, DOA, etc.), incorporating mobile receivers, and discretizing the signal space as a model of coarse/uncertain signal processing.**

***Index Terms*—sensor networks, opportunistic signals, mapping, localization**

I. Introduction

THIS article examines problems associated with localization, disambiguation, and mapping via ambient and uncontrolled signal sources of opportunity. In particular, we consider the case in which a mobile sensor/receiver collects relatively coarse signal data from several transmitters, perhaps to perform inference or reconstruction *a posteriori*. The (one or perhaps several) mobile sensor does not communicate with the rest of the network during the collection time and has little-to-no geographic data about its environment. We are motivated by minimal-sensing scenarios with a general lack of localization, as exemplified in (1) underground or underwater receivers, (2) multistatic radar disambiguation, or (3) adversarial or covert operations, in which the avoidance of detectable communication (or even self-interference) is paramount. We begin with a continuum approximation to the problem, develop the appropriate topological/geoemtric tools in this setting, then quantize the sensing and localization data along with the solution in a manner that is functorial — it respects the sensing operations.

A. An elementary example

We illustrate the themes of this paper with a trivial example. Consider a compact connected line segment $\mathcal{D} \subset \mathbb{R}$. Assume that there exist N transmitters in $\mathbb{R}$ of fixed location which asynchronously emit a pulse whose time of arrival can be measured by a receiver at any location in $\mathcal{D}$. To what extent do the received TOA (time of arrival) signals uniquely localize the receiver? Consider the signal profile mapping $\mathcal{P} : \mathcal{D} \to \mathbb{R}^N$ that records the TOA of the N (received, identified, and ordered) transmitter pulses. Clearly, this map $\mathcal{P}$ is continuous, and a generic perturbation of this map embeds the interval $\mathcal{D}$ into the "signals space" $\mathbb{R}^N$ for $N > 2$. Thus, a generic choice of signal profile mapping $\mathcal{P}$ is generically injective, implying unique channel response and the feasibility of localization in $\mathcal{D}$ via TOA.

This trivial observation is greatly generalizable to more arbitrary domains $\mathcal{D}$, encoding both physical and temporal data. In addition, one may modify the signals space to record different signals received from the transmitters. For example, DOA (direction of arrival) records bearing data; TDOA (time difference of arrival) reduces the dimension of the TOA signals space by one. More generally, one may build a transmitted signals space $\mathcal{S}$ from the individual transmitters and then reduce $\mathcal{S}$ by some modulation (a quotient map Φ) to a received signals domain $\mathcal{R}$. For example, the quotient of TOA signals $\mathcal{S} = \mathbb{R}^N$ by the action of the symmetric group S_N — as a model of the receiver's inability to identify target sources — yields as $\mathcal{R}$ a singular polytope.

The key insight of this paper is that replacing TOA or TDOA with *any* reasonable transmitter signal space $\mathcal{S}$ and modulation $\Phi : \mathcal{S} \to \mathcal{R}$ to a received signals space of sufficient dimension preserves the ability to localize: the induced SIGNAL PROFILE MAP: $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ will be generically injective as a function of dimension alone.

There are at least two unrealistic assumptions in this simplistic model: global signal propagation and infinite precision. In realistic environments, power limitations, obstacles, and directed transmissions conspire to limit the geometric extent of signal transmissions. This has the deleterious effect of making the signal profile mapping discontinuous, perhaps highly so in the setting of many transmitters. Our response is to construct a subcover of the domain on which a stable set of signals is heard. These *stable* regions are then pieced together to give a received signal profile mapping $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ which can be analyzed via differential-topological methods. The second problem addressed is that of accurate signal measurements: a poor assumption in many circumstances. We discretize the codomain of the signal profile map $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ and prove that a sufficiently fine discretization of $\mathcal{R}$ provides a reasonably discretized approximation to the image of $\mathcal{D}$. Coarser discretizations are permissible, and this yields very suggestive results for inference from minimal sensing.

B. Statement of results

We assume a compact domain $\mathcal{D}$ for receivers and a *generic* (to be specified carefully) set of transmitters providing a signals profile as per the assumptions of §II-C. Under these assumptions:

1) We prove a sufficient condition for receiver localization based on opportunistic signals as a function of the

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number and stable coverage of transmitters. This *Signals Embedding Theorem* is independent of the signal type (TOA, TDOA, DOA, etc.) and transmitter identification and permits fusion of multiple signal types.
2) Given a discretization of the signals into a finite alphabet of geometrically small cells, we prove that the code words as perceived by the receiver localize up to small cells. The diameters of these cells tend to zero as the partition of the signal domain is refined.
3) We verify our results experimentally using simple acoustic sounders. This demonstrates conclusively the feasibility of implementing these results in sensing contexts. Indeed, the equipment used for our experiment, though far from sufficient for the purpose of SONAR ranging or imaging, performs well for the task of detecting a change in the topology of the domain and validating our assumptions about the signal profile. Our experimental results also indicate the amount of signal corruption such a localization system can tolerate.

We emphasize that, as the methods employed are topological, the inferences possible are likewise topological, as opposed to rigidly geometric. Changes in the topology of a domain over time (How many buildings/obstacles are present? Is the window open?) are by no means unimportant.

C. Related work

There are many applications in which the signal sources are either unknown or uncontrolled, and yet their localizaton within the environment is still important. One of the most direct applications of this idea is the World-Wide Lightning Localization Network [1], which locates lightning strikes on the earth to within a few kilometers. This system uses a collection of radio sensors distributed on the earth's surface to correlate lightning strike arrival times. That such a distributed network can perform localization tasks under a variety of error models has been more extensively addressed in [2], [3].

Applications of opportunistic remote sensing are copious, as it is advantageous to exploit existing signal sources in the environment rather than create additional ones. Knowing a sources' position, power level, and waveform can greatly expedite its exploitation. Even in the best situation, in which the source and receiver locations are known, the required processing can be complicated [4]. Many of these algorithms consist of lifting traditional radar processing algorithms into a more general framework, such as Fourier transform methods [5], [6], time reversal [7], [8], equalization [9], [10], or Green's function approaches [11], [12].

Most experimental applications of opportunistic sensing in radar have focused on the use of large, publicly-recorded signal sources, such as digital broadcasters [13], [14], [15], [16], [17]. When such sources are not available, researchers have turned to the development of elaborate receivers with highly directive, steerable antennae [18]. We take a different approach, focusing on simple, inexpensive acoustic sounders that permit controlled experiments to be run in a laboratory setting.

These existing solutions suffer from a number of inherent limitations. Most evidently, they require intimate knowledge of source or receiver location and configuration. Additionally, they cannot reliably handle *multipath* (reflections, refractions, or diffractions of signals) without generating inconsistent results. Decontamination of multipath signals from a single additional scatterer was introduced in [19] and [20], with the definite understanding that this is a very limited case. More substantial multipath has the added *benefit* that if it can be correctly characterized, it can provide additional illumination for obstructed targets. When the multipath-generating scatterers are known, a filtered backprojection approach can be effective [21], [22], [23].

Our approach addresses some of the less-exercised aspects of the opportunistic sensing problem, by embracing both uncertainty (in locations and configuration of sources and receivers), and by exploiting multipath indirectly. Our methods consider the mapping from receiver location to the vector of signals received from each transmitter at these locations. When this mapping preserves the topology of the environment, it remains to interrogate the image of the mapping under sufficient sampling. Since the image lies in a (potentially) high dimensional signal space, the interrogation process can be thought of as a dimension reduction problem.

Indeed, estimation of the dimension [24], [25] of the environment from this kind of signal space mapping is an interesting problem, though it appears that there is no treatment of our particular mapping in the literature. Once the dimension is known, a number of algorithms related to nonlinear compressive sensing [26], [27], [28], [29], [30], [31] play a useful role in our analysis. In particular, they suggest that a random projection of the data to the appropriate dimension can result in an accurate *geometric* picture of the signal space. These kind of results are reminiscent of the Nash embedding theorem [32]. However, it is worth cautioning the reader that geometric information may be irreversibly lost depending on the sensing modality. In this case, an approach like that of [33] permits *detection* of certain features without complete recovery of the environment.

Our approach differs from the methods discussed previously in several important ways:
1) Our emphasis is on recovering topological features of the environment via signals of opportunity. To this end, we validate the experimental results presented in this article by computing topological invariants (persistent homology [34], [35]) of the domain as represented in the signal space and comparing them against ground truth.
2) We do not expect source or receiver locations to be known, and so focus on algorithms that are specifically non-coherent. As a benefit, algorithms developed in this framework (such as [36]) will be capable of working against poorer quality data than otherwise tolerable.
3) Our methods are robust with respect to discontinuities in received signals (near the minimum detectible signal levels) and also with respect to multipath contamination of the received signals.
4) Our framework addresses a wide array of known sensor modalities, such as those based on signal strength, time difference of arrival, direction of arrival, and more; mixed modalities are fully supported.

Finally, we address concerns about the feasibility of our approach by presenting experimental results that validate our signal model and the correctness of the resulting signal space embeddings.

II. Signals and signal profiles

In this section, we work in a continuum limit that permits a receiver to be located at any point in space: we will later sample this continuum by a network of fixed (or perhaps mobile) receivers in our discussion of our experimental results in §VI. All receivers reside in a compact domain $\mathcal{D}$ which is a manifold with boundary and (perhaps) corners (see Appendix for definitions from differential topology). It is common to visualize $\mathcal{D}$ as the physical workspace in which the transmitters and receivers reside, but this is not strictly necessary. For example, receivers with directional bias can be topologized as a space of cones in a tangent bundle; or, in the case where mobile receivers travel through a domain $A \subset \mathbb{R}^2$, the appropriate $\mathcal{D}$ may be the product $\mathcal{D} = A \times [0, T]$ with the time interval.

A. Spaces and signals

Received signals vary greatly. Among the most common received signal types are:

1) **TOA:** Time of arrival;
2) **TDOA:** Time difference of arrival;
3) **DOA:** Direction of arrival;
4) **Strength:** Strength of signal;
5) **Doppler:** Frequency difference of arrival.

The last item, doppler, requires that the transmitters and receivers are in motion relative to each other.

Having fixed the receiver domain $\mathcal{D}$ as a topological manifold in which receivers may be located, the next step is to topologize the set of possible signals, both transmitted and received, into signal spaces. In the simplest case (e.g., TOA with infinite signal propagation), a single received signal is a mapping $\mathcal{D} \to \mathbb{R}$ which records the time-of-arrival of the transmitted signal to a point $x \in \mathcal{D}$. More generally, a signal may take values in a manifold, such as $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, as in the case of DOA in the plane, or $\mathbb{R}^N$ in the case of $N+1$ transmitted signals under TDOA. Manifolds with corners or singularities are also possible, as in the case of TOA from multiple indistinguishable transmitters; the received signals lie in the polyhedral quotient of $\mathbb{R}^N$ by the action of the symmetric group S_N.

Consider, therefore, each transmitter to have an associated signals space $\mathcal{S}_i$. We encode limited signal range by means of a distinguished disjoint failstate basepoint $\perp$ which connotes failure to receive this signal. For example, the TOA signal space for N finite-range transmitters would be $\mathcal{S}_i = \mathbb{R} \sqcup \perp$ with the full signals space being the cross product $\mathcal{S} = (\mathbb{R} \sqcup \perp)^N$; an element of this space is an ordered N-tuple of times-of-arrival of the N identified signals, with $\perp$ meaning that the signal either failed to arrive or was of insufficient SNR. The topology on $\mathbb{R} \sqcup \perp$ is, as implied, disjoint union.

B. The transmission and signal profiles

The connection between the receiver domain $\mathcal{D}$ and the spaces of signals induced by signal transmission and reception takes the form of mappings. We differentiate between the transmitted signals readable by a receiver and the information that a receiver retains, perhaps after signal modulation. For example, in TDOA, individual transmitter signals are detected by the receiver; however, only the time-difference between incoming signals is retained as received signal data. We encode this difference of readable and retained signals by means of a quotient map $\Phi : \mathcal{S} \to \mathcal{R}$ between the space of transmitted signals $\mathcal{S}$ and received or retained signals $\mathcal{R}$. A receiver at a point in $\mathcal{D}$ receives a transmission signal by means of a transmission profile map $\mathcal{T} : \mathcal{D} \to \mathcal{S}$ and an induced received signal profile map $\mathcal{P} : \mathcal{D} \to \mathcal{R}$, where $\mathcal{P} = \Phi \circ \mathcal{T}$.

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\mathcal{T}"] \arrow[dr, "\mathcal{P}"'] & \mathcal{S} = \prod_i (\mathcal{S}_i \sqcup \perp) \arrow[d, "\Phi"] \\ & \mathcal{R} \end{tikzcd} \quad . \tag{1}$$

C. Assumptions

For the remainder of the paper, we enforce the following axiomatic characterization of signal profiles:

1) $\mathcal{D}$ is a manifold with boundary and corners.
2) Each transmitter emits a signal which is reliably readable by a receiver in $\mathcal{D}$ on a STABLE DOMAIN $U_i \subset \mathcal{D}$, a compact codimension-0 submanifold with corners.
3) Each transmitter determines a smooth TRANSMISSION MAP $\mathcal{T}_i \in C^\infty(U_i, \mathcal{S}_i)$ taking values in a TRANSMISSION SIGNAL SPACE $\mathcal{S}_i$.
4) Each transmission map extends to $\mathcal{T}_i : \mathcal{D} \to \mathcal{S}_i \sqcup \perp$ and evaluates to $\perp$ on points outside of U_i.
5) The individual signal maps assemble into the TRANSMISSION PROFILE, the map $\mathcal{T} : \mathcal{D} \to \mathcal{S} = \prod_i (\mathcal{S}_i \sqcup \perp)$ given by the product of the $\mathcal{T}_i$ maps.
6) The SIGNAL PROFILE $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ is the postcomposition of the transmission profile $\mathcal{P}$ with a quotient map $\Phi : \mathcal{S} \to \mathcal{R}$, where $\mathcal{R}$ is a disjoint union of manifolds and Φ is a submersion (the derivative $d\Phi$ is onto at each point of $\mathcal{S}$).

As an example, for TDOA with infinite broadcast range ($U_i = \mathcal{D}$ for $1 \leq i \leq N$), the quotient map Φ from the transmission signal space $\mathcal{S} = \mathbb{R}^N$ to the received time-difference space $\mathcal{R} = \mathbb{R}^{N-1}$ is a linear projection map (time-difference) with $d\Phi$ of constant rank $N-1$ everywhere.

III. The Signals Embedding Theorem

We demonstrate that for sufficiently many generic transmission signals, each point in $\mathcal{D}$ has a unique signal profile. The critical resource is the number (and dimension) of signals received relative to $\dim \mathcal{D}$. The collection of stable domains for the transmitters is denoted $\mathscr{U} = \{U_i\}_1^N$. It will be assumed that $\mathscr{U}$ is a cover for $\mathcal{D}$, meaning that the union of the interiors of the U_i sets contains $\mathcal{D}$. We characterize the amount of information needed to uniquely localize receivers via signals in terms of a depth of the collection of stable domains $\mathscr{U}$.

Definition 1. Given a domain $\mathcal{D}$ and a cover $\mathscr{U}$ of $\mathcal{D}$ by sets $\mathscr{U} = \{U_\alpha\}$, the DEPTH of the cover, $\operatorname{dep} \mathscr{U}$, is the minimal $n \in \mathbb{N}$ such that every point $x \in \mathcal{D}$ lies in at least n distinct elements of $\mathscr{U}$.

Definition 2. Given $\mathcal{P} : \mathcal{D} \to \mathcal{R}$, consider the localization $\mathcal{T}_x$ of $\mathcal{T}$ taking a neighborhood of $x \in \mathcal{D}$ to only those signals $\mathcal{S}_i$ for which x lies in the *interior* of U_i. Define $\mathcal{P}_x = \Phi \circ \mathcal{T}_x$ and declare the $\mathcal{P}$-weighted depth $\mathcal{P}$-WEIGHTED DEPTH $\operatorname{dep} \mathcal{P}$ of the cover $\mathscr{U}$ is defined as the minimal rank of the derivatives $d\mathcal{P}_x$ at x over all x:

$$\operatorname{dep} \mathcal{P} = \min_n \{\operatorname{rank}\, d\mathcal{P}_x \geq n \;\forall\, x \in \mathcal{D}\}\,. \tag{2}$$

The received signal profile $\mathcal{P}$ may or may not be injective. When it is not, receivers at different locations record identical signals. It may be the case that such ambiguity is an extreme coincidence, and a small perturbation to the individual signals removes the ambiguity in $\mathcal{P}$. On the other hand, non-uniqueness of signal profiles may be a persistent feature of the environment: although a perturbation may alter signal values at two specific receiver locations, nearby receiver locations will, after the perturbation, become identified. Our principal result specifies the degree of possible ambiguity.

Theorem 3 (Signals Embedding Theorem)**.** *Let $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ be a received signal profile with stable domains $\mathscr{U} = \{U_i\}_1^N$ satisfying the assumptions of §II-C. For generic transmitters — specifically, for individual transmission signal maps $\mathcal{T}_i$ open and dense in $C^\infty(U_i, \mathcal{S}_i)$ — the set of points in $\mathcal{D}$ on which $\mathcal{P}$ is non-injective is of dimension*

$$\begin{aligned}\dim \{x \in \mathcal{D} : \mathcal{P}(x) = \mathcal{P}(y) \text{ for some } y \neq x\} \\ \leq 2 \dim \mathcal{D} - \operatorname{dep} \mathcal{P}.\end{aligned}$$

Proof: Begin with the following assumptions: (1) all transmissions are of unbounded extent ($U_i = \mathcal{D}$ for all i), so that $\mathcal{S} = \prod_i \mathcal{S}_i$; and (2) the quotient map Φ is the identity, so that $\mathcal{P} : \mathcal{D} \to \mathcal{R} = \mathcal{S}$. In this case, the signal profile $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ is globally smooth and $\operatorname{dep} \mathcal{P} = \dim \mathcal{S}$. The result flows from the following version of the Whitney Embedding Theorem. A generic perturbation of the transmission signal maps $\mathcal{T}_i$ is equivalent to a generic perturbation of the received signal profile $\mathcal{P}$, since the topologies on $C^\infty(\mathcal{D}, \prod_i \mathcal{S}_i)$ and $\prod_1^N C^\infty(\mathcal{D}, \mathcal{S}_i)$ are equivalent [37]. Consider the configuration space,

$$\begin{aligned}\mathcal{C}^2\mathcal{D} &= \mathcal{D} \times \mathcal{D} - \Delta_{\mathcal{D}} \\ \Delta_{\mathcal{D}} &= \{(x, y) \in \mathcal{D} \times \mathcal{D} \,:\, x = y\}\end{aligned}$$

of two distinct points on $\mathcal{D}$. This is a manifold (with corners, as per $\mathcal{D}$) of dimension $2 \dim \mathcal{D}$. The graph of the signal profile $\mathcal{P}$ induces a map on the configuration space:

$$\begin{aligned}\mathcal{C}^2\mathcal{P} &: \mathcal{C}^2\mathcal{D} \to \mathcal{C}^2\mathcal{D} \times \mathcal{S} \times \mathcal{S} \\ &: (x, y) \mapsto (x, y, \mathcal{P}(x), \mathcal{P}(y)).\end{aligned}$$

The set of points on which $\mathcal{P}$ is non-injective is precisely $(\mathcal{C}^2\mathcal{P})^{-1}(\mathcal{C}^2\mathcal{D} \times \Delta_{\mathcal{S}})$, where $\Delta_{\mathcal{S}} \subset \mathcal{S} \times \mathcal{S}$ is the diagonal. According to the multi-jet transversality theorem , cf. [38, Thm. 4.13], generic perturbations of $\mathcal{P}$ induce generic perturbations of $\mathcal{C}^2\mathcal{P}$ (since $\mathcal{C}^2\mathcal{P}$ is the 2-fold 0-jet of $\mathcal{P}$). Thus, from transversality and the inverse mapping theorem, the generic dimension of the non-injective set equals:

$$\begin{aligned}\dim \mathcal{C}^2\mathcal{D} + \dim \mathcal{C}^2\mathcal{D} \times \Delta_{\mathcal{S}} - \dim \mathcal{C}^2\mathcal{D} \times \mathcal{S} \times \mathcal{S} \\ = 2 \dim \mathcal{D} + 2 \dim \mathcal{D} + \dim \mathcal{S} - (2 \dim \mathcal{D} + 2 \dim \mathcal{S}) \\ = 2 \dim \mathcal{D} - \dim \mathcal{S} = 2 \dim \mathcal{D} - \operatorname{dep} \mathcal{P}.\end{aligned}$$

The transversality theorems invoked — both the multi-jet transversality and inverse mapping theorems — are usually stated for maps between smooth manifolds without boundary; however, they apply also in the case of a manifold with corners [39]. For a compact domain $\mathcal{D}$, as is here the case, the stronger conclusion of *open, dense* instead of *generic* holds [38, Prop. 5.8]. This completes the proof for the case $U_i = \mathcal{D}$ for all i and $\Phi = \mathrm{Id}$.

Next, relax the assumptions on the quotient map $\Phi : \mathcal{S} \to \mathcal{R}$ from being an identity to being a submersion — the derivative $d\Phi$ is everywhere of full rank equal to $\dim \mathcal{R}$. Then, following the initial case, we wish to perform perturbations in the transmission signals $C^\infty(\mathcal{D}, U_i)$, while controlling the injectivity of $\Phi \circ \mathcal{T}$. The non-injective set of $\mathcal{P}$ equals the inverse image

$$(\mathcal{C}^2\mathcal{P})^{-1}\left(\mathcal{C}^2\mathcal{D} \times (\Phi \times \Phi)^{-1}(\Delta_{\mathcal{R}})\right),$$

where $\Delta_{\mathcal{R}} \subset \mathcal{R} \times \mathcal{R}$ is the diagonal. As Φ is a submersion, the inverse image of $\Delta_{\mathcal{R}}$ under the product map $\Phi \times \Phi$ is of dimension $2 \dim \mathcal{S} - \dim \mathcal{R}$. Thus, from transversality and the inverse mapping theorem, the generic dimension of the non-injective set equals:

$$\begin{aligned}= 2 \dim \mathcal{D} + (2 \dim \mathcal{D} + 2 \dim \mathcal{S} - \dim \mathcal{R}) \\ -(2 \dim \mathcal{D} + 2 \dim \mathcal{S}) \\ = 2 \dim \mathcal{D} - \dim \mathcal{R} = 2 \dim \mathcal{D} - \operatorname{dep} \mathcal{P}.\end{aligned}$$

This completes the proof in the case of globally-received signals $U_i = \mathcal{D}$.

Finally, we relax to limited range signals. Assume a tame cover $\mathscr{U} = \{U_i\}_1^N$ of $\mathcal{D}$ of stable domains for transmission signal reception. The intersection lattice of $\mathscr{U}$ consists of all nonempty intersections of elements of $\mathscr{U}$. Let $\mathscr{V}$ denote the collection of closures of the elements of this intersection lattice: by tameness, $\mathscr{V}$ is again a finite cover of $\mathcal{D}$ by compact codimension-0 submanifolds of $\mathcal{D}$ with corners. For convenience, use a multi-index $J \in \{\pm 1\}^N$ for $\mathscr{V} = \{V_J\}$ encoded so that

$$V_J = \operatorname{closure}\left(\bigcap_{J_i = +1} U_i \;\cap\; \bigcap_{J_k = -1} (\mathcal{D} - U_k)\right).$$

On each (nonempty) V_J, restricting $\mathcal{T}$ to the positive-J coordinates and ignoring the $\perp$ states on the remaining coordinates yields a smooth map $\mathcal{T}_J : V_J \to \prod_{J_i = +1} \mathcal{S}_i$. Further restriction of Φ yields a $\mathcal{P}_J : V_J \to \mathcal{R}_J$ that is a map between smooth manifolds. By assumption on the $\mathcal{P}$-weighted depth, the rank of $d\mathcal{P}_J$ (restricted to the interior) is at least $\operatorname{dep} \mathcal{P}$. Thus, as per the previous case, for an open dense set of transmission signal maps, the dimension of the non-injective set of $\mathcal{P}_J$ is bounded above by $2 \dim \mathcal{D} - \operatorname{dep} \mathcal{P}$. Repeating the argument for each multi-index J and taking the finite intersection of the resulting open dense sets of transmission signal maps completes the proof. ■

IV. Corollaries

The Signals Embedding Theorem provides simple criteria for the signal *depth* required to ensure a generic injection into the received signals space. If the dimension of the self-intersection set of $\mathcal{P}$ is negative, then $\mathcal{P}$ is one-to-one, and its image in the received signals space $\mathcal{R}$ is a faithful image of $\mathcal{D}$, partitioned according to $\mathscr{U}$.

A. Depth criteria for localization

In many instances, depth criteria for unique channel response is a function of the depth of the cover $\mathscr{U}$ by stable signal domains.

Corollary 4. *A generic TOA, signal strength, or range signal profile is injective whenever* $\text{dep } \mathscr{U} > 2 \dim \mathcal{D}$.

Proof: In the case of TOA, signal strength, or range reception, each signal space $\mathcal{S}_i = \mathbb{R}$, $\Phi = \text{Id}$, and $\text{dep } \mathcal{P} = \text{dep } \mathscr{U}$. From Theorem 3, the subset of $\mathcal{D}$ on which $\mathcal{P}$ is generically non-injective is of negative dimension — hence empty — when $\text{dep } \mathscr{U} > 2 \dim \mathcal{D}$. ■

This implies that a receiver can be localized to a unique position in a planar domain $\mathcal{D}$ using only a sequence of *five* or more locally stable TOA/strength/range readings from generic transmitters. For TDOA, six signals are required to achieve localization:

Corollary 5. *A generic TDOA signal profile is injective whenever* $\text{dep } \mathscr{U} > 2 \dim \mathcal{D} + 1$.

Proof: Each $\mathcal{S}_i = \mathbb{R}$ and the reduction map $\Phi : \mathcal{S} \to \mathcal{R}$ is a submersion of rank defect one; hence $\text{dep } \mathcal{P} = \text{dep } \mathscr{U} - 1$. ■

DOA has a more dramatic effect on required signal depths.

Corollary 6. *A generic DOA signal profile is injective whenever*

$$\text{dep } \mathscr{U} > \frac{2 \dim \mathcal{D}}{\dim \mathcal{D} - 1}. \tag{3}$$

Proof: Each $\mathcal{S}_i = \mathbb{S}^{\dim \mathcal{D} - 1}$ and $\Phi = \text{Id}$. Thus, for injectivity,

$$2 \dim \mathcal{D} < \text{dep } \mathcal{P} = (\text{dep } \mathscr{U})(\dim \mathbb{S}^{\dim \mathcal{D} - 1}).$$

■

Remark 7. Note that Corollary 6 implies that

1) DOA localization is impossible when $\dim \mathcal{D} = 1$;
2) for a planar domain, there is no difference between DOA and TOA in terms of signal cover depth required; and
3) for domains of dimension greater than three, the signal depth required for DOA equals *three*, independent of dimension.

B. Time-dependent systems

1) Pulses: Consider the setting in which transmitters and receivers are collocated in a physical domain $\mathcal{D}$, and the transmitters are in motion over a time interval $[0, T]$. One simple instantiation of this would be the following. Consider a single transmitter which moves through $\mathcal{D}$, emitting a pulse signal intermittently over the time interval $[0, T]$. Each pulse is audible over a stable domain U_i and induces a signal in $\mathcal{S} = (\mathcal{S}_i \sqcup \perp)^N$, where N is the number of pulses emitted and all Σ_i are the same (under TOA, TDOA, or DOA). Assuming that perturbation of the motion of the transmitter and the propagation of pulses induces a generic perturbation of the transmission profile map $\mathcal{T} : \mathcal{D} \to \mathcal{S}$, Theorem 3 and Corollaries 4-6 immediately translate to the case of a single (mobile) transmitter sending multiple pulses.

2) Path-crossing: Other time-dependent settings are more interesting. Consider the space-time product $\mathcal{D} = \mathcal{D}' \times [0, T]$ and the case of one or more transmitters in motion in $\mathcal{D}'$. Then, in the setting of TOA, TDOA, DOA, signal strength, or range, one has a cover of $\mathcal{D}$ by stable patches U_i delineating where and when a signal is readable. By Theorem 3, the signal profile map $\mathcal{P} : \mathcal{D}' \times [0, T] \to \mathcal{R}$ is injective for $\text{dep}(\mathcal{P}) > 2(\dim \mathcal{D}' + 1)$. Note well that this localizes *temporally* as well as spatially. This injectivity criterion has additional potential utility. Assume that two mobile *receivers* move through the domain $\mathcal{D}$ over the time interval $[0, T]$, but have no information about their locations or relative distances. Did the receivers ever cross paths?

Corollary 8. *Two mobile receivers moving along paths* $\gamma_1, \gamma_2 : [0, T] \to \mathcal{D}$ *intersect in* $\mathcal{D}$ *if and only if their images in* $\mathcal{R}$ *under* $\mathcal{P}$ *intersect, assuming* $\text{dep}(\mathcal{P}) > 2 \dim \mathcal{D}$.

Note that this applies to $\mathcal{D}$ a physical or a space-time domain, allowing for determination of whether the receivers covered the same territory at *some* times or whether the receivers actually met. That such inference may be rigorously concluded *a posteriori* within the received signal space $\mathcal{R}$ seems novel. The reader may easily derive other similar generalizations for inference via received signals.

3) Doppler: One last aspect of dynamic transmitter-receiver systems involves sensing via doppler. In this case, receivers can measure both the range-to-transmitted (from TOA plus synchronization) and the doppler shift in the received signal. The doppler shift takes values in $\mathbb{R}$, since there is an increase in the received frequency for an approaching transmitter and a decrease in the received frequency for a retreating one. In the context of multiple transmitters, synchronization across all platforms is difficult. As a result, it is more realistic to measure TDOA plus doppler for the received signals. The following corollary follows trivially.

Corollary 9. *A generic TDOA-plus-doppler signal profile localizes in time and space over a domain* $\mathcal{D}$ *whenever*

$$\text{dep } \mathscr{U} > \dim \mathcal{D} + 3/2. \tag{4}$$

Proof: Each $\mathcal{S}_i = [0, \infty) \times \mathbb{R}$ and Φ has rank defect one. Thus, for injectivity,

$$2 \dim \mathcal{D} + 2 < \text{dep } \mathcal{P} = 2N - 1.$$

■

C. Anonymous transmissions

The following asserts that anonymization of transmitter sources does not impact signal depth criteria.

Proposition 10. *For a signal profile in which all transmission signals are of the same type ($\mathcal{S}_i = X$ for all i) and the quotient map to $\mathcal{R}$ is equivariant with respect to transmitter identities (Φ is invariant under the action of the symmetric group S_N on $\mathcal{S}$), then passing from identified to unidentified transmitters does not change the dimension bounds on the self-intersection set in Theorem 3.*

Proof: In our formulation, the transmission profile demands signal identities, since $\mathcal{T} : \mathcal{D} \to \prod_i (X \sqcup \perp)$. The action of the symmetric group $S_N : \mathcal{S} \to \mathcal{S}$ permuting transmitter identities descends by equivariance to an action $S_N : \mathcal{R} \to \mathcal{R}$. The quotient $\mathcal{R} \to \mathcal{R}/S_N$ is not a submersion, as the action of S_N is not free, as in the case of two signals arriving at the same time, in which the non-identity permutation of the transmitter identities is a fixed point. However, the action of S_N *is* free (and therefore has derivative of full rank) on a dense codimension-0 submanifold (the complement of the Φ-image of the pairwise diagonal in X^N). The dimension bounds in the proof of Theorem 3 are sensitive only to top-dimensional phenomena; thus, $\operatorname{dep} \mathcal{P}$ remains unchanged after quotienting by the S_N action. ∎

D. Multi-modal sensing and fusion

Since genericity can be decoupled to each of the transmission functions independently, there is no reason why each should represent the same kind of signal modality. In contrast to some of the examples in the previous sections, we could well consider a heterogeneous family of transmitters. For instance, consider the situation where there are N transmitters for whom the receivers can detect signal strength only, but there are M for which signal strength and doppler can be measured. In this case, $\mathcal{S} = (\mathbb{R} \sqcup \perp)^N \times (\mathbb{R}^2 \sqcup \perp)^M$, which leads to $\operatorname{dep} \mathcal{P} = N + 2M$. A consequence of this situation is that one can imagine design constraints that balance the availability of inexpensive transmitters with more expensive (but more capable) transmitters. The reader may easily generalize.

E. Configuration spaces

There is no reason why $\mathcal{D}$ must conform to a physical or even physical-temporal locus of receivers. Consider the dual setting in which N receivers are fixed at locations in a physical domain X (a compact manifold with corners). A collection of M transmitters operate at distinct locations in X. The parameter space (to be embedded in a signals space) is the space of configurations of the M (labeled) transmitters, $\mathcal{C}^M(X) = X^M - \Delta_X$, where $\Delta_X = \{x_i = x_j$ for some $i \neq j\}$. Let $\mathcal{D} = \mathcal{C}^M(X)$, where, to ensure compactness, one removes a sufficiently small open neighborhood of the pairwise diagonal Δ_X. For simplicity, consider the restricted case where all N receivers can hear all M transmitters, and that the received signals are scalar-valued, as in TOA/strength/range settings. The following result uses Theorem 3 to derive the existence of triangulation-without-distance algorithms: one can triangulate position based on a non-isotropic signal without knowledge of locations or actual distance.

Corollary 11. *Under the above assumptions, the transmitter positions are unambiguous for $N > 2\dim X$, independent of M.*

Proof: From Theorem 3, the criterion is:

$$2M \dim X = 2\dim(C^M(X)) < \dim \mathcal{S} = MN.$$

∎

In particular, for a planar domain, the positions of the transmitters is uniquely encoded in signal space by *five* fixed receivers, independent of the number of (audible) transmitters, providing a dual to Corollary 4. Five exceeds the three needed for triangulation of position via geometry: for weaker topological signals, more data is required. We leave it to the reader to work out the appropriate statements for localizing transmitters uniquely when the signals have only partial extent or operate under different modalities than range/TOA/strength or the receiver/transmitter identities are anonymous.

F. Design spaces

Theorem 3 can be applied to settings that go well beyond the motivating scenario of receivers localizing based on transmission signals. Consider the following (idealized) setting. A dish antenna is characterized by a small collection of parameters: radius of curvatures in principal directions of the reflector, principal diameters of the reflector, and the location and orientation of feed. As a result, each dish antenna's space of possible designs is a manifold of dimension 10. Of course, many other possible design spaces are possible, and the idea readily handles other antenna configurations, such as electrically-scanned phased arrays.

In this setting, we therefore let $\mathcal{D}$ correspond to the design space for the antenna. As before, $\mathcal{D}$ will be a finite-dimensional manifold, perhaps with boundary or corners. The boundaries and corners of $\mathcal{D}$ correspond to the imposition of design rules and constraints that are common to the kind of antenna under examination.

The mapping $\mathcal{T} : \mathcal{D} \to \mathcal{S}$ is given by the placement of N receivers at fixed points in the propagation field. For $\mathbb{R}$-valued reception signals and globally stable transmissions, $\mathcal{S} = \mathbb{R}^N = \mathcal{R}$ and the Signals Embedding Theorem declares how many readings of the antenna are required to unambiguously determine its design parameters.

It should be noted that in this case, each of the transmission functions $\mathcal{T}_i$ are not independent. Changing any portion of the antenna's design will generally result in a modification of *all* of the $\mathcal{T}_i$ in a possibly complicated fashion. It is therefore unnecessary to consider perturbing each independently, so one could use the standard Whitney embedding theorem for manifolds with corners instead.

Corollary 12. *For $\mathcal{D}$ the finite-dimensional design space (manifold with corners) of an antenna, the antenna characteristics are uniquely determined by sampling at N generic points in the propagation domain, where $N > 2\dim \mathcal{D}$.*

In the case of more complicated antennas, for instance electrically scanned phased arrays, we remind the reader that $\dim \mathcal{D}$ may be quite large. This is a reflection of the fact that the elemental excitations and element patterns may be unknown, and therefore introduced as parameters in $\mathcal{D}$. This causes no theoretical problem, but collection of sufficient data may present a challenge. This is actually rather striking as the usual approach to measuring phased arrays *requires* $\mathbb{C}$-valued measurements to be taken. Although the approach outlined here requires the same amount of data (possibly in more measurements), there is *no* requirement on the format of the data.

It should also be noted that Corollary 12 presents a highly efficient way to measure antenna radiation patterns. Rather than sampling the entire antenna pattern, which can be extremely time-consuming, one need only sample enough to measure the antenna's design in $\mathcal{D}$. From this solution, the remainder of the antenna's pattern can be *predicted* including where it has not been directly measured.

V. Quantization

Discretized signals would seem to promise effective localization. However, a straightforward application of Theorem 3 fails: using a quotient map Φ from $\mathcal{S}$ to a finite set (of dimension zero) yields a $\mathcal{P}$-weighted depth $\operatorname{dep} \mathcal{P} = 0$. Clearly, a quantized signal profile cannot be injective; however, if the quantization is fine enough, quantized signals should distinguish points in $\mathcal{D}$ up to some small distance. We begin by specifying a geometry on $\mathcal{S}$: suppose that each $\mathcal{S}_i$ is a Riemannian manifold, with induced metric d_i. The metric on $\mathcal{S}$ is the product metric on the d_i, with the (intrinsic) convention that d takes on the value ∞ if the points are in distinct connected components of $\mathcal{S}$. Furthermore, $\mathcal{R}$ is likewise assumed to have Riemannian metrics on components. The following compactness result indicates the feasibility of metric quantization.

Proposition 13. *Let $\mathcal{P} : \mathcal{D} \to \mathcal{R}$ be a received signal profile with stable domains $\mathscr{U} = \{U_i\}_1^N$ satisfying the assumptions of §II-C, with, in addition: (1) $\mathcal{S}$ and $\mathcal{R}$ are Riemannian on connected components; and (2) $\operatorname{dep} \mathcal{P} > 2 \dim \mathcal{D}$. For individual transmission signal maps $\mathcal{T}_i$ open and dense in $C^\infty(U_i, \mathcal{S}_i)$, and for $\epsilon > 0$ small, there exists a constant $K(\epsilon) > 0$ such that:*

$$\operatorname{diam} \mathcal{P}^{-1}\left(B_\epsilon(\mathcal{P}(x))\right) < K(\epsilon)$$

uniformly in $x \in \mathcal{D}$, with $\lim_{\epsilon \to 0^+} K(\epsilon) = 0$.

Proof: Recall from the proof of Theorem 3 the cover $\mathscr{V} = \{V_J\}$ of $\mathcal{D}$ by compact closures of the intersection lattice of the stable sets $\mathscr{U}$. From the hypothesis on $\operatorname{dep} \mathcal{P}$, the restriction of $\mathcal{P}$ to each V_J (with the canonical extension to any added boundary components) is a smooth embedding of V_J onto its image. Smoothness and compactness yields a function $K_J(\epsilon)$ bounding the diameters of preimages of the restriction. As $\mathscr{V}$ is finite, there is a uniform $K(\epsilon)$ for which the result holds. ∎

This result indicates that a quantization on the received signals space yields an ambiguity in the domain $\mathcal{D}$ of bounded size; this in itself is suboptimal, since the bounds might be poor.

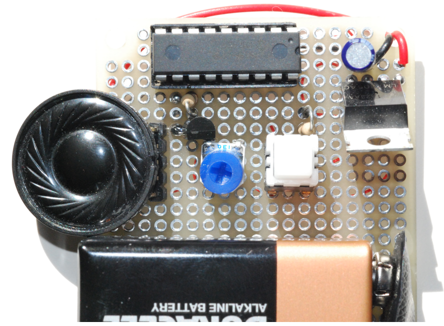

Fig. 1. Acoustic sounders were constructed to be transmitters in experiments.

VI. Experimental validation

To compensate for lack of hard bounds on quantization ambiguity, and to test the applicability of the Signals Embedding Theorem, we constructed a simple experiment with acoustical sounders. The platform-independence of Theorem 3 gives considerable freedom in selecting the transmitted waveforms and the construction of the experiment. Since we focus on the topological characterization of a propagation domain in this article, we conducted an experiment to demonstrate:

1) The correctness of our our axoimatic characterization of the signal profile;
2) That the condition number of the resulting signal profile is positive, which validates the feasibility of our theory, and furthermore;
3) That it is possible to detect a change in the homotopy type of the domain by means of the signal profile measured at a collection of receiver locations.

Several transmitters were constructed (see Figure 1) from a PIC16F88 microcontroller, a simple audio pre-amplifier, and a speaker. The microcontroller runs custom firmware that causes the sounder to emit square waves with arbitrary transition times. For the experiments discussed here, the transmitters were commanded to emit chirped pulses in the range of 5kHz-10kHz. Signal reception was accomplished by the use of a standard laptop computer sound card. The computer ran a custom real-time matched filter bank (using the GStreamer multimedia framework) tuned to each of the transmitters. When triggered by the user, the computer stores the magnitude of each matched filter tap in a datafile for later processing.

The experiment was conducted on a laboratory floor cleared of acoustically reflective obstacles in the immediate vicinity. Scatterers were present outside of the experimental area, resulting in potential multipath returns. For each run of the experiment, transmitters were placed at fixed locations (labeled 1-4), and the receiver was raster-scanned throughout the experimental domain (avoiding any obstacles) with a spacing of 3 inches between samples. For two of the runs, an acoustically opaque obstacle (a stack of books) was placed within the experimental volume. See Figure 2 for details of the layout and Table I for a listing of the experimental runs. Runs B and C collectively consider the case where there is an obstacle in the domain (and so the domain is an annulus), while D and E address the case when the domain is contractible.

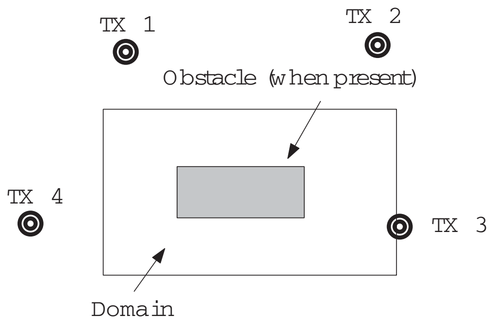

Fig. 2. Spatial layout of the experiment.

TABLE I
LISTING OF EXPERIMENTAL RUNS.

Run	Obstacle	Transmitters	Comments
A	No	1, 2	Calibration run (not shown)
B	Yes	1, 3	Experimental collection
C	Yes	2, 4	Experimental collection
D	No	1, 3	Experimental collection
E	No	2, 4	Experimental collection

A. Validation of signal model

The received signal levels corresponding to each transmitter are displaced in Figure 3, which incorporates a choice of signal level threshold (independently for each transmitter) to simulate the failure of reception and some spatial filtering to compensate for receiver instability. It is immediately clear from these plots that there is a stable domain containing each transmitter, and that away from this domain the reception becomes erratic before dropping out completely. We regard this as a validation of assumption (2) of the signal profile.

Given the experimentally collected data, it is straightforward to infer properties of the signal profile. In this experiment, the quantized signal profile was injective, in that each receiver location had a unique response to the set of transmitters. Given the potentially large dynamic range of the data, we found that 18 dB of signal-to-noise ratio was required to ensure that the resulting quantized signal profile remains injective. Figure 4 shows the lowest signal-to-noise ratio required to maintain injectivity at a given receiver location, and indicates that this is fairly stable over the domain with an average value of roughly 15 dB for both sets of runs.

The maximal depth of the cover in this experiment is *three*, which is below the required (*five*) for guaranteed signal profile injectivity. However, even this appears to suffice for the purposes of detecting the difference in topological type of the domains used in runs B,C versus D,E. To see this, first consider the projection of the data into two dimensions given in Figure 5. As indicated by the compressive sensing literature, a random projection ought to result in a reasonably accurate picture of the environment embedded in the signal space, but the dimension of the signal space is too low for the asymptotic results to be of much use. Using the formulae in [28] with the appropriate values from the experiment results in a likelihood of about 10% that a random projection will be sufficiently close to an isometry to be topologically accurate.

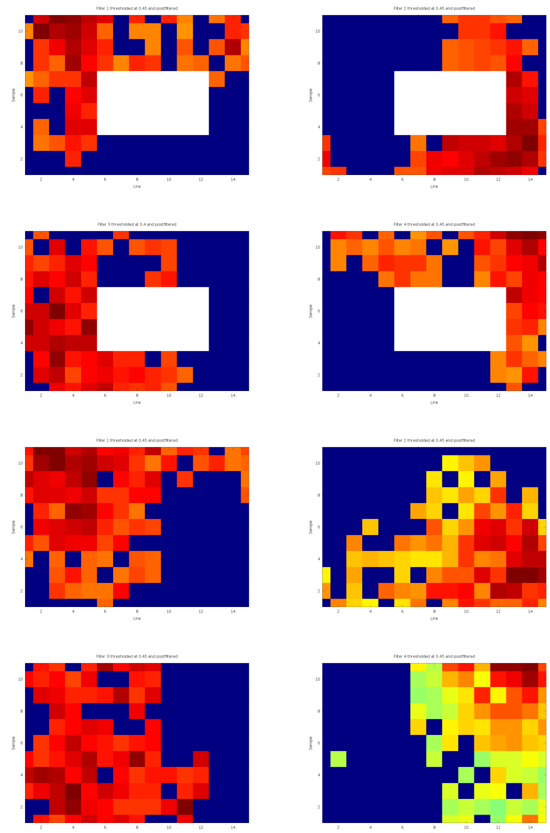

Fig. 3. Thresholded signal levels from each transmitter as a function of position. The top four frames represent runs B,C. Missing portions of the data correspond to the presence of the obstacle.

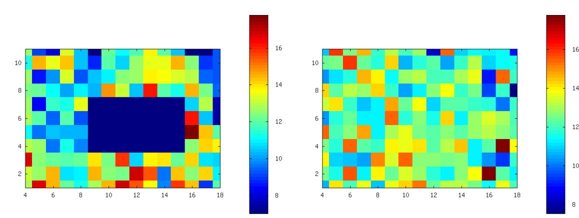

Fig. 4. Signal-to-noise ratio required to maintain injectivity of the signal profile at a given receiver location in runs B,C (left) and runs D,E (right)

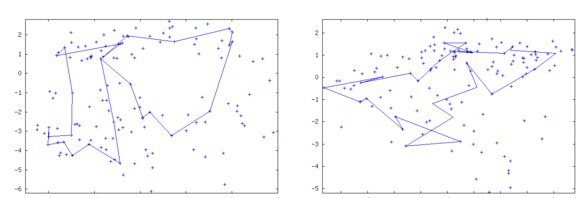

Fig. 5. Projection of received signal levels at each receiver. The data from runs D,E (right) have been randomly downsampled to match the number of points in runs B,C (left). The marked path tightly bounds the obstacle, when it is present.

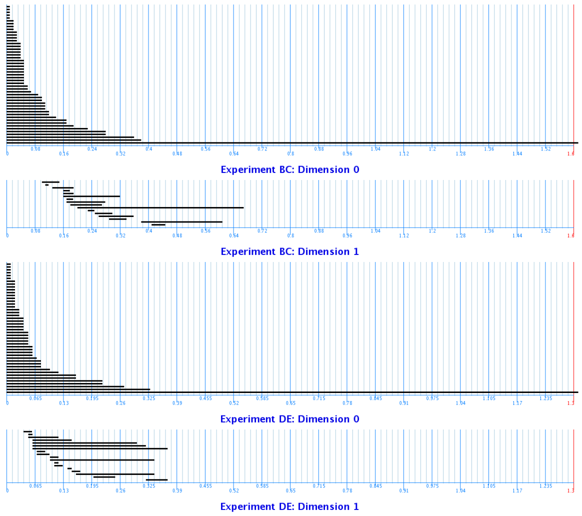

Fig. 6. Persistent homology barcodes of the two collection runs. The horizontal axis represents signal level in dB.

Evidently the projection in Figure 5 is far from random. Instead, we have plotted a particularly illuminating projection. In this projection, one observes that the image of the domain through the signal profile seems correct.

This observation requires luck in choosing a projection: a more objective measure of validity is desirable. Recently, PERSISTENT HOMOLOGY has emerged as an effective tool for examining the topology of point clouds sampled from a topological space [34], [35]. This algebraic method discriminates between a contractible planar domain and one punctured by obstacles. In particular, the presence of a persistent generator of homology in dimension one (H_1) indicates the presence of an obstacle. We computed this persistent homology using `JPlex` [40]; its signature, or BARCODE, is shown in Figure 6. It is immediately clear that both runs came from connected domains (since there is one dimension 0 that persists for almost all scales). There are some persistent generators in dimension 1 for both sets of runs, but there is only one substantial generator in experiments B and C that persists in excess of 0.5 dB. This indicates the strong possibility of the presence of a 1-dimensional hole in the domain for the case of runs B and C and not in runs D and E. Hence, we conclude that the experiment has detected a topological change in the domain, and in particular identifies the presence of one obstacle in runs B and C and no obstacles in D and E.

VII. CONCLUSION

This paper builds a general theoretical framework in which to analyze signals of opportunity and utilize such to characterize a domain in terms of a representation into the appropriate space of signals. Of note in our approach are the following features:

1) Instead of trying to reconstruct coordinates within the domain, it can be effective and profitable to work completely within the space of signals. Given sufficient control over the signal profile depth, this representation is faithful, modulo the discontinuities induced by limited-extent signals.
2) One advantage of working within a space of signals is the independence of the signal type. The topological approach reveals that dimension is the critical resource for faithful representation.
3) Although the differential-topological tools used assume a high degree of regularity and ignore noise and other inescapable system features, the robustness of the results to quantization — as verified in theory, simulation, and experiment — argues for wide applicability.

ACKNOWLEDGMENT

This work supported by DARPA STO - HR0011-09-1-0050, DARPA DSO - HR0011-07-1-0002 and ONR N000140810668.

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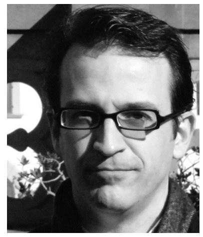

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APPENDIX

A DIFFERENTIAL n-MANIFOLD is a paracompact Hausdorff space M with an open covering $\mathscr{U} = \{U_\alpha\}$ and maps $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ which are homeomorphisms onto their images and for which the restriction of $\phi_\beta\phi_\alpha^{-1}$ to $U_\alpha \cap U_\beta$ is a smooth (C^∞ for our purposes) diffeomorphism whenever $U_\alpha \cap U_\beta \neq \emptyset$. One says that an n-manifold is MODELED on $\mathbb{R}^n$ via CHARTS U_α in an ATLAS $\mathscr{U}$. An n-manifold with BOUNDARY is a space locally modeled on $\mathbb{R}^n$ or the upper halfspace $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, depending on the chart. An n-manifold with CORNERS allows a choice of any $(\mathbb{R}^+)^k \times \mathbb{R}^{n-k}$ as local models.

To each point p in a manifold M is associated a TANGENT SPACE, T_pM, a $\mathbb{R}$-vector space of dimension $\dim M$ that records tangent data at p. The collection of tangent spaces fit together into a TANGENT BUNDLE, a manifold T_*M, defined locally as charts of M crossed with $\mathbb{R}^{\dim M}$. Maps between manifolds are said to be smooth if the restriction of the map to charts yields smooth maps between charts. Such maps $f : M \to N$ induce a DERIVATIVE $Df : T_*M \to T_*N$ defined on charts via the Jacobian derivative. The JET BUNDLE $J^r(M, N)$ is the manifold which records all degree r Taylor polynomials associated to maps in $C^r(M, N)$, with the topology inherited from M (source points), N (target points), and the usual topology on real coefficients of polynomials. We use C^∞ smoothness in this paper, and place the usual (Whitney) C^∞ topology on the space $C^\infty(M, N)$ of smooth maps from M to N: a C^∞ neighborhood of $f : M \to N$ has basis functions g whose r-jets are close, as measured by the topology on $J^r(M, N)$.

A subset $A \subset X$ is RESIDUAL if it is the countable intersection of open dense subsets of X. For BAIRE spaces, like $C^\infty(M, N)$, residual sets are always dense. A property is GENERIC (or holds generically) with respect to a parameter space if that property is true on a residual subset of the parameter space.

Two submanifolds V, W in M are transverse, written $V \pitchfork W$, if and only if $T_pV \oplus T_pW = T_pM$ for all $p \in V \cap W$ — the tangent spaces to V and W span that of Mat intersections. Note that the absence of intersection is automatically transverse. A smooth map $f : V \to M$ to a submanifold $W \subset N$ if and only if $Df_v(T_vV) \oplus (T_pW) = T_pM$ whenever $f(v) = p$. The JET TRANSVERSALITY THEOREM states that for W a submanifold of $J^r(M, N)$, the set of maps in $C^\infty(M, N)$ whose r-jets are transverse to W is residual. This readily yields the simpler transversality theorem that the subset of $C^\infty(M, N)$ transverse to a submanifold $W \subset N$ is residual (and, furthermore, open if W is closed).